Convergence of the block Jacobi methods

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OUTLINE

- Introduction
- Pivot strategies
- Block Jacobi operators
- Convergence results
- 4×4 matrices
- Applications

INTRODUCTION

- Jacobi method is an iterative process for solving matrix eigenproblem $\mathbf{A}x = \lambda x$, $x \neq 0$, where $\mathbf{A}$ is a symmetric matrix.

- One step of the method:

$$\mathbf{A}^{(k+1)} = \mathbf{U}_k^T \mathbf{A}^{(k)} \mathbf{U}_k, \quad k \geq 0, \quad \mathbf{A}^{(0)} = \mathbf{A},$$

  where $\mathbf{U}_k = R(i(k), j(k), \phi(k))$ are orthogonal matrices.

- Goal of the k-th step:

  make $\mathbf{A}^{(k+1)}$ "more diagonal" than $\mathbf{A}^{(k)}$.

- Reducing the **off-norm** of $\mathbf{A}$:

$$S^2(\mathbf{A}) = \frac{\|\mathbf{A} - \mathrm{diag}(\mathbf{A})\|_F^2}{2} = \sum_{i<j} a_{ij}^2$$

INTRODUCTION

- **Convergence:** $\mathbf{A}^{(k)} \to \mathbf{\Lambda}$, $\mathbf{\Lambda}$ diagonal matrix
- We show that

$$S(\mathbf{A}') \leq \gamma S(\mathbf{A}), \quad 0 \leq \gamma < 1,$$

where $\mathbf{A}'$ is the matrix obtained from $\mathbf{A}$ after one full cycle and constant γ does not depend on $\mathbf{A}$.

HISTORY

- Jacobi (1846) studies the relationships between seven planets of the Solar system. He does not use matrices since matrix as an object was introduced several years later, in 1850 by Sylvester.
- The method only became widely used in the 1950s with the advent of computers.
- First papers studying the convergence theory are from 1960s (Forsythe, Henrici, Hansen).

- C. G. J. Jacobi: *Über ein leichtes Verfahren, die in der Theorie der Säkularstörungen vorkommenden Gleichungen numerisch aufzulösen*. Crelle's Journal 30 (1846) 51–96.

HISTORY

- Over the last two decades the Jacobi method has emerged as a method of choice for the eigenvalue computation for dense symmetric matrices. This is mostly due to its inherent **parallelism** and **high relative accuracy** on well-behaved matrices

- J. Demmel, K. Veselić: *Jacobi's method is more accurate than QR*. SIAM J. Matrix Anal. Appl. 13 (4) (1992) 1204–1245.
- Z. Drmač, K. Veselić: *New fast and accurate Jacobi SVD algorithm I*. SIAM J. Matrix Anal. Appl. 29 (4) (2008) 1322–1342.
- Z. Drmač, K. Veselić: *New fast and accurate Jacobi SVD algorithm II*. SIAM J. Matrix Anal. Appl. 29 (4) (2008) 1343–1362.

HISTORY

"In the Mathematics Department in Zagreb in 1969, Krešimir Veselić was appointed to teach Numerical Analysis, a very new topic at that time. (...) For the final exam, Egon Zakrajšek was allowed to pick his own topic and present it to the class. It was a brilliant presentation of the classical Jacobi method for computing eigenvalues of symmetric matrices, then completely unknown to Croatian researchers."

from Z. Drmač: *Linear Algebra in Croatia: Spiritus Movens and Curious Events*. ILAS IMAGE 49 (2012) 6–10.

ELEMENT-WISE METHOD

$$\mathbf{A}^{(k+1)} = \mathbf{U}_k^T \mathbf{A}^{(k)} \mathbf{U}_k, \quad k \geq 0, \quad \mathbf{A}^{(0)} = \mathbf{A},$$

- Rotation matrices:

$$\mathbf{U} = R(i,j,\phi) = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\phi & & & & -\sin\phi & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & \sin\phi & & & & \cos\phi & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} \end{array}$$

BLOCK JACOBI METHOD

- $\pi = (n_1, n_2, \ldots, n_m)$ is a partition of n,

$$n_1 + n_2 + \cdots + n_m = n, \quad n_i \geq 1, \quad 1 \leq i \leq m.$$

- $\mathbf{A}^{(0)} = \mathbf{A}$ is a symmetric square $n \times n$ **block matrix**,

$$\mathbf{A} = \begin{array}{c} \left[\begin{array}{cccc} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{array} \right] \end{array} \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_m \end{array}$$

BLOCK JACOBI METHOD

- $\mathbf{U}_k$, $k \geq 0$ are orthogonal **elementary block matrices**,

$$\mathbf{U}_k = \mathbf{U}_{i(k),j(k)} = \left[\begin{array}{ccccc} I &&&& \\ & U_{ii} && U_{ij} & \\ && I && \\ & U_{ji} && U_{jj} & \\ &&&& I \end{array}\right] \begin{array}{c} \\ n_i \\ \\ n_j \\ \\ \end{array}$$

- Matrices

$$\hat{\mathbf{U}}_{ij} = \left[\begin{array}{cc} U_{ii} & U_{ij} \\ U_{ji} & U_{jj} \end{array}\right] \quad \text{and} \quad \hat{\mathbf{A}}_{ij} = \left[\begin{array}{cc} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{array}\right]$$

are **pivot submatrices** and $(i,j) = (i(k), j(k))$ is **pivot pair**.

- $\mathbf{U}_{ij} = \mathcal{E}(i, j, \hat{\mathbf{U}}_{ij})$

BLOCK JACOBI METHOD

k-th step:

$$\begin{bmatrix} \Lambda_{ii}^{(k+1)} & 0 \\ 0 & \Lambda_{jj}^{(k+1)} \end{bmatrix} = \begin{bmatrix} U_{ii}^{(k)} & U_{ij}^{(k)} \\ U_{ji}^{(k)} & U_{jj}^{(k)} \end{bmatrix}^T \begin{bmatrix} A_{ii}^{(k)} & A_{ij}^{(k)} \\ (A_{ij}^{(k)})^T & A_{jj}^{(k)} \end{bmatrix} \begin{bmatrix} U_{ii}^{(k)} & U_{ij}^{(k)} \\ U_{ji}^{(k)} & U_{jj}^{(k)} \end{bmatrix}$$

PREPROCESSING

$$
\mathbf{A} = \left[\begin{array}{ccc|cc|c|cc}
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x
\end{array}\right]
$$

$$\downarrow$$

$$
\mathbf{A}^{(0)} = \left[\begin{array}{ccc|cc|c|cc}
x & 0 & 0 & x & x & x & x & x \\
0 & x & 0 & x & x & x & x & x \\
0 & 0 & x & x & x & x & x & x \\
\hline
x & x & x & x & 0 & x & x & x \\
x & x & x & 0 & x & x & x & x \\
\hline
x & x & x & x & x & x & x & x \\
\hline
x & x & x & x & x & x & x & 0 \\
x & x & x & x & x & x & 0 & x
\end{array}\right]
$$

UBC MATRICES

- **Uniformly bounded cosine (UBC) matrices** $\mathbf{U}_k = (U_{rs}^{(k)})$ (Drmač, 2009)

$$\min_{1 \le r \le m} \sigma_{min}(U_{rr}^{(k)}) \ge c > 0, \quad k \ge k_0 \ge 0.$$

- This is generalization of the Forsythe-Henrici condition for the serial element-wise Jacobi method:

  The cosine of the rotation angles has to be uniformly bounded by a positive constant.

- Elementary UBC matrices are denoted by UBCE.

Pivot strategies

PIVOT STRATEGIES

- **Pivot strategy** can be identified with a function

$$I : \mathbb{N}_0 \to P_m,$$

where $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $P_m = \{(i, j) \mid 1 \leq i < j \leq m\}$.

- **Periodic** pivot strategy with period T
- **Cyclic**: if $T = M \equiv \frac{m(m-1)}{2}$ and $\{I(k) \big| 0 \leq k \leq M-1\} = P_m$
- **Quasi-cyclic**: if $T \geq M$ and $\{I(k) \big| 0 \leq k \leq M-1\} = P_m$

PIVOT ORDERINGS

- By $\mathcal{O}(P_m)$ we denote the set of all finite sequences containing the elements of P_m, assuming that each pair from P_m appears at least once in each sequence.
- For a given cyclic strategy I, **pivot ordering** is the sequence

$$\mathcal{O}_I = (i(0), j(0)), \ldots, (i(M-1), j(M-1)) \in \mathcal{O}(P_m).$$

- For a given ordering $\mathcal{O} \in \mathcal{O}(P_m)$, the cyclic strategy $I_{\mathcal{O}}$ is defined by

$$I_{\mathcal{O}}(k) = (i_{\tau(k)}, j_{\tau(k)}), \qquad 0 \leq \tau(k) \leq M-1,$$

where $k \equiv \tau(k) (\text{mod } M)$, $k \geq 0$.

PIVOT ORDERINGS

- Visualization of an ordering $\mathcal{O}$ of P_m:

  symmetric $m \times m$ matrix $\mathrm{M}_{\mathcal{O}} = (\mathrm{m}_{rt})$ such that

$$\mathrm{m}_{i(k)j(k)} = \mathrm{m}_{j(k)i(k)} = k, \quad k = 0, 1, \ldots, M-1.$$

  We set $\mathrm{m}_{rr} = -1$, $1 \leq r \leq m$.

- Example: Serial pivot orderings, column-wise and row-wise

$$\mathrm{M}_{\mathcal{O}_c} = \begin{bmatrix} -1 & 0 & 1 & 3 & 6 \\ 0 & -1 & 2 & 4 & 7 \\ 1 & 2 & -1 & 5 & 8 \\ 3 & 4 & 5 & -1 & 9 \\ 6 & 7 & 8 & 9 & -1 \end{bmatrix}, \ \mathrm{M}_{\mathcal{O}_r} = \begin{bmatrix} -1 & 0 & 1 & 2 & 3 \\ 0 & -1 & 4 & 5 & 6 \\ 1 & 4 & -1 & 7 & 8 \\ 2 & 5 & 7 & -1 & 9 \\ 3 & 6 & 8 & 9 & -1 \end{bmatrix}.$$

EQUIVALENT ORDERINGS

- **Admissible transposition** on $\mathcal{O} \in \mathcal{O}(\mathcal{S})$, $\mathcal{S} \subseteq P_m$ is a transposition of two adjacent terms

$$(i_r, j_r), (i_{r+1}, j_{r+1}) \to (i_{r+1}, j_{r+1}), (i_r, j_r),$$

provided that $\{i_r, j_r\}$ and $\{i_{r+1}, j_{r+1}\}$ are disjoint.

EQUIVALENT ORDERINGS

Two sequences $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(P_m)$ are:

(i) **equivalent** (we write $\mathcal{O} \sim \mathcal{O}'$) if one can be obtained from the other by a finite set of admissible transpositions,

$$(4,6),(5,6),(1,2),(1,3) \sim (4,6),(1,2),(5,6),(1,3)$$

(ii) **shift-equivalent** ($\mathcal{O} \overset{\mathsf{s}}{\sim} \mathcal{O}'$) if $\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2]$ and $\mathcal{O}' = [\mathcal{O}_2, \mathcal{O}_1]$, where [,] stands for concatenation,

$$(4,6),(5,6),(3,6),(4,5) \overset{s}{\sim} (3,6),(4,5),(4,6),(5,6)$$

(iii) **weak equivalent** ($\mathcal{O} \overset{\mathsf{w}}{\sim} \mathcal{O}'$) if there exist $\mathcal{O}_i \in \mathcal{O}(P_m)$, $0 \leq i \leq r$, such that in the sequence

$$\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r = \mathcal{O}'$$

every two adjacent terms are equivalent or shift-equivalent.

CONVERGENT ORDERINGS

Theorem (Hansen 1963, Shroff and Schreiber 1989)

If a block Jacobi method converges for some cyclic ordering, then it converges for all orderings that are weak equivalent to it.

The block methods defined by equivalent cyclic orderings generate the same matrices after each full cycle and within the same cycle they produce the same sets of orthogonal elementary matrices.

What do we do?

Enlarge the set of "convergent orderings".

INVERSE ORDERINGS

- Let $\mathcal{O} \in \mathcal{O}(P_m)$, $\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{M-1}, j_{M-1})$. Then

$$\mathcal{O}^{\leftarrow} = (i_{M-1}, j_{M-1}), \ldots, (i_1, j_1), (i_0, j_0) \ \in \mathcal{O}(P_m)$$

  is **inverse ordering** to $\mathcal{O}$.

- Obviously, $\mathcal{O}^{\leftarrow\leftarrow} = \mathcal{O}$.

PERMUTATION EQUIVALENT ORDERINGS

- Two pivot orderings $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(P_m)$ are **permutation equivalent** if
$$\mathsf{M}_{\mathcal{O}'} = \mathsf{P}\mathsf{M}_{\mathcal{O}}\mathsf{P}^T$$
holds for some permutation matrix P. We write $\mathcal{O}' \overset{\mathrm{p}}{\sim} \mathcal{O}$.
- One cycle of the Jacobi method on matrix A with pivot strategy $\mathbf{I}_{\mathcal{O}}$ generates the same matrix as one cycle on matrix $P^T AP$ with strategy $\mathbf{I}_{\mathcal{O}'}$.

Block Jacobi operators

FUNCTION vec

- Column vector of $\mathbf{X} \in \mathbb{R}^{p \times q}$

$$\text{col}(\mathbf{X}) = [x_{11}, x_{21}, \ldots, x_{p1}, \ldots, x_{1q}, \ldots, x_{pq}]^T$$

- Let $\mathbf{A}$ be a symmetric block matrix with partition $\pi = (n_1, \ldots, n_m)$. Then

$$\text{vec}_\pi(\mathbf{A}) = \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{where } c_j = \begin{bmatrix} \text{col}(A_{1j}) \\ \text{col}(A_{2j}) \\ \vdots \\ \text{col}(A_{j-1,j}) \end{bmatrix}, \ 2 \leq j \leq m.$$

- $\text{vec}_\pi : \mathbf{S}_n \to \mathbb{R}^K, \qquad K = N - \sum_{i=1}^{m} \frac{n_i(n_i-1)}{2}, \ N = \frac{n(n-1)}{2},$

  where $\mathbf{S}_n$ is the space of symmetric $n \times n$ matrices, is a **linear operator**.

FUNCTION vec_0 and OPERATOR $\mathcal{N}_{ij}$

- Let $\mathbf{S}_{0,n}$ be the space of symmetric $n \times n$ matrices whose diagonal blocks are zero. Then $\mathrm{vec}_0 = \mathrm{vec}|_{\mathbf{S}_{0,n}}$ is **bijection**.
- Linear operator $\mathcal{N}_{ij} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ "diagonalizes" the pivot submatrix of the argument matrix,

$$\mathcal{N}_{ij}(\mathbf{X}) = \begin{bmatrix} X_{11} & & \cdots & & X_{1m} \\ & \mathrm{diag}(X_{ii}) & & 0 & \\ \vdots & & & & \vdots \\ & 0 & & \mathrm{diag}(X_{jj}) & \\ X_{m1} & & \cdots & & X_{mm} \end{bmatrix}.$$

BLOCK JACOBI ANNIHILATORS

Definition

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n. Let $\hat{\mathbf{U}} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix}$, be orthogonal $(n_i + n_j) \times (n_i + n_j)$ matrix and let $\mathbf{U} = \mathcal{E}(i, j, \hat{\mathbf{U}})$ be the appropriate elementary block matrix.

The transformation $\Im_{ij}(\hat{\mathbf{U}})$ determined by

$$\Im_{ij}(\hat{\mathbf{U}})(vec(\mathbf{A})) = vec(\mathcal{N}_{ij}(\mathbf{U}^T \mathbf{A} \mathbf{U})), \qquad \mathbf{A} \in \mathbf{S}_n$$

is called *ij*-**block Jacobi annihilator**.

For each pair $1 \leq i < j \leq m$,

$$\Im_{ij} = \left\{ \Im_{ij}(\hat{\mathbf{U}}) \;\middle|\; \hat{\mathbf{U}} \text{ is orthogonal matrix of order } n_i + n_j \right\}$$

is the *ij*-**class of block Jacobi annihilators**.

Computing $\Im_{ij}(\hat{U})a$

$a \in \mathbb{R}^K$ an arbitrary vector
$A = vec_0^{-1}(a)$

// $\mathbf{A}$

FOR $r = 1, \ldots, m$
  $A'_{ri} = A_{ri}U_{ii} + A_{rj}U_{ji}$
  $A'_{rj} = A_{ri}U_{ij} + A_{rj}U_{jj}$
ENDFOR

// $\mathbf{U}^T\mathbf{AU}$

FOR $r = 1, \ldots, m$
  $A'_{ir} = U_{ii}^T A_{ir} + U_{ji}^T A_{jr}$
  $A'_{jr} = U_{ij}^T A_{ir} + U_{jj}^T A_{jr}$
ENDFOR

$A'_{ij} = 0$, $A'_{ji} = 0$
$A'_{ii} = \mathrm{diag}(A'_{ii})$, $A'_{jj} = \mathrm{diag}(A'_{jj})$

// $\mathcal{N}_{ij}(\mathbf{U}^T\mathbf{AU})$

$a' = vec(A')$

BLOCK JACOBI ANNIHILATORS - Example

Let $\mathbf{A} \in \mathbb{R}^{8\times 8}$, $\pi = (2,2,2,2)$, $i = 1$, $j = 2$. Then $K = 24$ and

$$
\mathfrak{S}_{12}(\hat{\mathbf{U}}) = \left[\begin{array}{c|c|c|c|c|c}
\begin{matrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{matrix} & & & & & \\ \hline
 & \begin{matrix} U_{11}^T & \\ & U_{11}^T \end{matrix} & \begin{matrix} U_{21}^T & \\ & U_{21}^T \end{matrix} & & & \\ \hline
 & \begin{matrix} U_{12}^T & \\ & U_{12}^T \end{matrix} & \begin{matrix} U_{22}^T & \\ & U_{22}^T \end{matrix} & & & \\ \hline
 & & & \begin{matrix} U_{11}^T & \\ & U_{11}^T \end{matrix} & \begin{matrix} U_{21}^T & \\ & U_{21}^T \end{matrix} & \\ \hline
 & & & \begin{matrix} U_{12}^T & \\ & U_{12}^T \end{matrix} & \begin{matrix} U_{22}^T & \\ & U_{22}^T \end{matrix} & \\ \hline
 & & & & & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix}
\end{array}\right],
$$

where U_{11}, U_{12}, U_{21}, U_{22} are 2×2 blocks of $\hat{\mathbf{U}} \in \mathbb{R}^{4\times 4}$ and $\hat{\mathbf{U}}$ is orthogonal.

BLOCK JACOBI ANNIHILATORS

- $\|\Im_{ij}(\hat{\mathbf{U}})\|_2 = 1$,
  except for $m = 2$ when $\Im_{12}(\hat{\mathbf{U}}) = 0$.
- $\mathfrak{S}$ differs from the identity matrix I_K in exactly $m-1$ principal submatrices.
- $\mathfrak{S} \in \boldsymbol{\mathfrak{S}}_{ij} \;\Rightarrow\; \mathfrak{S}^T \in \boldsymbol{\mathfrak{S}}_{ij}$
- $\mathfrak{S} \in \boldsymbol{\mathfrak{S}}_{ij}^{\mathrm{UBC}} \;\Rightarrow\; \mathfrak{S}^T \in \boldsymbol{\mathfrak{S}}_{ij}^{\mathrm{UBC}}$
- Henrici, Zimmerman 1968; Hari 2009, 2015; BK 2014

BLOCK JACOBI ANNIHILATORS

Transforming a matrix process into a vector process

$$\mathbf{A}^{(k+1)} = \mathcal{N}_{ij}(\mathbf{U}_k^T \mathbf{A}^{(k)} \mathbf{U}_k)$$
$$\downarrow$$
$$a^{(k+1)} = \Im_{i(k)j(k)}(\hat{U}_k) a^{(k)}$$

BLOCK JACOBI OPERATORS

Definition

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n and

$$\mathcal{O} = (i_0, j_0), (i_1, j_1), \ldots, (i_{M-1}, j_{M-1}) \in \mathcal{O}(P_m).$$

Then

$$\mathcal{J}_{\mathcal{O}} = \{\mathcal{J} \mid \mathcal{J} = \mathfrak{S}_{i_{M-1}j_{M-1}} \mathfrak{S}_{i_{M-2}j_{M-2}} \ldots \mathfrak{S}_{i_0 j_0}, \\ \mathfrak{S}_{i_k j_k} \in \boldsymbol{\mathfrak{S}}_{i_k j_k},\ 0 \leq k \leq M-1\}$$

is the class of block Jacobi operators associated with ordering $\mathcal{O}$. The $K \times K$ matrices $\mathcal{J}$ from $\mathcal{J}_{\mathcal{O}}$ are **block Jacobi operators**.

BLOCK JACOBI OPERATORS - Properties

If two sequences $\mathcal{O}, \mathcal{O}' \in \mathcal{O}(S)$ are

- **equivalent** $\;\Rightarrow\; \mathcal{J}_{\mathcal{O}} = \mathcal{J}_{\mathcal{O}'}$,
- **shift-equivalent** $\;\Rightarrow\; \mathrm{spr}(\mathcal{J}_{\mathcal{O}}) = \mathrm{spr}(\mathcal{J}_{\mathcal{O}'})$,
- **weak equivalent** $\;\Rightarrow\; \mathrm{spr}(\mathcal{J}_{\mathcal{O}}) = \mathrm{spr}(\mathcal{J}_{\mathcal{O}'})$,
- **permutation equivalent** $\;\Rightarrow\; \|\mathcal{J}_{\mathcal{O}}\|_2 = \|\mathcal{J}_{\mathcal{O}'}\|_2$,
- **reverse to each other** $\;\Rightarrow\; \|\mathcal{J}_{\mathcal{O}}\|_2 = \|\mathcal{J}_{\mathcal{O}'}\|_2$.

Convergence results

SERIAL STRATEGIES WITH PERMUTATIONS

Examples:

$$\begin{bmatrix} * & 0 & 2 & 4 & 9 & 12 \\ 0 & * & 1 & 5 & 8 & 10 \\ 2 & 1 & * & 3 & 7 & 13 \\ 4 & 5 & 3 & * & 6 & 11 \\ 9 & 8 & 7 & 6 & * & 14 \\ 12 & 10 & 13 & 11 & 14 & * \end{bmatrix} \in \mathcal{B}_c^{(6)}, \qquad \begin{bmatrix} * & 11 & 13 & 12 & 10 & 14 \\ 10 & * & 9 & 7 & 6 & 8 \\ 11 & 9 & * & 5 & 3 & 4 \\ 12 & 6 & 5 & * & 1 & 2 \\ 13 & 7 & 3 & 1 & * & 0 \\ 14 & 8 & 4 & 2 & 0 & * \end{bmatrix} \in \mathcal{B}_r^{(6)}$$

SERIAL STRATEGIES WITH PERMUTATIONS

Formal definition:

$$\mathcal{B}_c^{(m)} = \{\mathcal{O} \in \mathcal{O}(P_m) \mid \mathcal{O} = (1,2),(\pi_3(1),3),(\pi_3(2),3),\ldots,$$
$$(\pi_m(1),m),\ldots,(\pi_m(m-1),m), \quad \pi_j \in \Pi^{(1,j-1)},\ 3 \le j \le m\},$$
$$\mathcal{B}_r^{(m)} = \{\mathcal{O} \in \mathcal{O}(P_m) \mid \mathcal{O} = (m-1,m),(m-2,\tau_{m-2}(m)),(m-2,\tau_{m-2}(m-1)),\ldots,$$
$$(1,\tau_1(m)),\ldots,(1,\tau_1(2)), \quad \tau_i \in \Pi^{(i+1,m)},\ 1 \le i \le m-2\},$$

where $\Pi^{(l_1,l_2)}$ stands for the set of all permutations of $\{l_1, l_1+1, \ldots, l_2\}$.

SERIAL STRATEGIES WITH PERMUTATIONS

$$\overleftarrow{\mathcal{B}}_c^{(m)} = \{\mathcal{O} \in \mathcal{O}(P_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_c\},$$
$$\overleftarrow{\mathcal{B}}_r^{(m)} = \{\mathcal{O} \in \mathcal{O}(P_m) \mid \mathcal{O}^{\leftarrow} \in \mathcal{B}_r\}.$$

Examples:

$$\begin{bmatrix} * & 14 & 12 & 10 & 5 & 2 \\ 14 & * & 13 & 9 & 6 & 4 \\ 12 & 13 & * & 11 & 7 & 1 \\ 10 & 9 & 11 & * & 8 & 3 \\ 5 & 6 & 7 & 8 & * & 0 \\ 2 & 4 & 1 & 3 & 0 & * \end{bmatrix} \in \overleftarrow{\mathcal{B}}_c^{(6)}, \qquad \begin{bmatrix} * & 4 & 3 & 2 & 1 & 0 \\ 4 & * & 5 & 8 & 7 & 6 \\ 3 & 5 & * & 9 & 11 & 10 \\ 2 & 8 & 9 & * & 13 & 12 \\ 1 & 7 & 11 & 13 & * & 14 \\ 0 & 6 & 10 & 12 & 14 & * \end{bmatrix} \in \overleftarrow{\mathcal{B}}_r^{(6)}$$

SERIAL STRATEGIES WITH PERMUTATIONS

Set

$$\mathcal{B}_{sp}^{(m)} = \mathcal{B}_c^{(m)} \cup \overleftarrow{\mathcal{B}_c}^{(m)} \cup \mathcal{B}_r^{(m)} \cup \overleftarrow{\mathcal{B}_r}^{(m)}.$$

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sp}^{(m)}$ and let $\mathcal{J} \in \mathcal{J}_{\mathcal{O}}^{\mathrm{UBC}}$ be a block Jacobi operator.

Then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that

$$\|\mathcal{J}\|_2 \leq \gamma_\pi, \quad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

SERIAL STRATEGIES WITH PERMUTATIONS

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sp}^{(m)}$ and let $\mathbf{A} \in \mathbf{S}_n$ be a block matrix. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by applying one sweep of the cyclic block Jacobi method defined by the strategy $I_{\mathcal{O}}$.

If all transformation matrices are from the class *UBCE*, then there are constants γ_π (depending only on π) and $\tilde{\gamma}_n$ (depending only on n) such that

$$S^2(\mathbf{A}') \leq \gamma_\pi S^2(\mathbf{A}), \qquad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

GENERALIZED SERIAL STRATEGIES

Set

$$\mathcal{B}_{sg}^{(m)} = \left\{ \mathcal{O} \in \mathcal{O}(P_m) \mid \mathcal{O} \overset{\mathrm{p}}{\sim} \mathcal{O}' \overset{\mathrm{w}}{\sim} \mathcal{O}'' \text{ or } \mathcal{O} \overset{\mathrm{w}}{\sim} \mathcal{O}' \overset{\mathrm{p}}{\sim} \mathcal{O}'',\ \mathcal{O}'' \in \mathcal{B}_{sp}^{(m)} \right\}.$$

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$ and let $\mathcal{J} \in \mathcal{J}_{\mathcal{O}}^{\mathrm{UBC}}$ be a block Jacobi operator. Suppose the chain connecting $\mathcal{O}$ to $\mathcal{O}'' \in \mathcal{B}_{sp}^{(m)}$ is in canonical form and contains d shift equivalences.

Then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that for any $d+1$ Jacobi operators $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_{d+1} \in \mathcal{J}_{\mathcal{O}}^{\mathrm{UBC}}$ it holds

$$\|\mathcal{J}_1 \mathcal{J}_2 \cdots \mathcal{J}_{d+1}\|_2 \leq \gamma_\pi, \quad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

GENERALIZED SERIAL STRATEGIES

Theorem (BK, Hari)

Let $\pi = (n_1, \ldots, n_m)$ be a partition of n, $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$ and let $\mathbf{A} \in \mathbf{S}_n$ be a block matrix. Suppose the chain connecting $\mathcal{O}$ and $\mathcal{O}''$ is in canonical form and contains d shift equivalences. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by applying $d+1$ sweeps of the cyclic block Jacobi method defined by the strategy $I_{\mathcal{O}}$.

If all transformation matrices are from the class *UBCE*, then there are constants γ_π and $\tilde{\gamma}_n$ depending only on π and n, respectively, such that

$$S^2(\mathbf{A}') \leq \gamma_\pi S^2(\mathbf{A}), \qquad 0 \leq \gamma_\pi < \tilde{\gamma}_n < 1.$$

4×4 matrices

"I start by looking at a 2×2 matrix. Sometimes I look at a 4×4 matrix. That's when things get out of control and too hard."
-Paul Halmos

4×4

We can assume that $I(0) = (1, 2)$.

$$\begin{bmatrix} * & 0 & x & x \\ x & * & x & x \\ x & x & * & x \\ x & x & x & * \end{bmatrix}$$

Then there are $5! = 120$ possible pivot strategies.

Out of those 120 startegies

$\rightarrow$ 104 strategies are from the class $\mathcal{B}_{sg}^{(4)}$

- 16 parallel strategies

$$M_{I_1} = \begin{bmatrix} * & 0 & 2 & 4 \\ 0 & * & 5 & 3 \\ 2 & 5 & * & 1 \\ 4 & 3 & 1 & * \end{bmatrix}$$

4×4

Theorem (BK, Hari)

Let $\mathbf{A} \in S_4$ be such that $a_{12} = 0$, $a_{34} = 0$, and let $\mathbf{A}^{(12)}$ be obtained by applying 12 steps of the Jacobi method under the strategy I_1 to $\mathbf{A}$. Then

$$S(\mathbf{A}^{(12)}) \leq (1 - \epsilon) S(\mathbf{A}),$$

with $\epsilon = 10^{-5}$.

Theorem (BK, Hari)

Let $\mathbf{A} \in S_4$, and let $\mathbf{A}^{[3]}$ be obtained by applying 3 cycles of the Jacobi method under any pivot strategy. Then there is a constant γ such that

$$S^2(\mathbf{A}^{[3]}) \leq \gamma S^2(\mathbf{A}), \quad 0 \leq \gamma < 1.$$

Applications

GENERAL JACOBI-TYPE PROCESS

Iterative process of the form

$$\mathbf{A}^{(k+1)} = \mathbf{F}_k^{-1}\mathbf{A}^{(k)}\mathbf{F}_k, \quad k \geq 0,$$

where $\mathbf{F}_k, k \geq 0$, are elementary block matrices.
Pivot submatrices of $\mathbf{F}$ are only required to be nonsingular.

Assumptions

A1: $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$;

A2: There is a sequence of orthogonal elementary block matrices $\mathbf{U}_k$, $k \geq 0$, such that

$$\lim_{k\to\infty}(\mathbf{F}_k - \mathbf{U}_k) = 0;$$

A3: $\mathbf{U}_k$ are UBCE matrices.

GENERAL JACOBI-TYPE PROCESS

Theorem

Let $\pi = (n_1, n_2, \ldots, n_m)$ be a partition of n, let $\mathbf{A} \in S_n$, $\mathbf{A} \neq \mathbf{0}$ be a block matrix with partition defined by π, and let $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$. Let the sequence of matrices $\mathbf{A}^{(k)}$, $k \geq 0$, be generated by a general block Jacobi-type process. If the assumptions **A1–A3** are met, then the following two assertions are equivalent:

(i) $$\lim_{k\to\infty} \frac{\text{off}\left(\hat{A}_{i(k)j(k)}^{(k+1)}\right)}{\|\mathbf{A}^{(k)}\|_F} = 0,$$

(ii) $$\lim_{k\to\infty} \frac{\text{off}\left(\mathbf{A}^{(k)}\right)}{\|\mathbf{A}^{(k)}\|_F} = 0.$$

GENERALIZED EIGENVALUE PROBLEM

Positive definite generalized eigenvalue problem (PGEP)

$$\mathbf{A}x = \lambda \mathbf{B}x, \quad x \neq 0,$$

where **A**, **B** are symmetric matrices and **B** is positive definite.

Convergence: $(\mathbf{A}, \mathbf{B}) \to (\mathbf{\Lambda}, \mathbf{I})$, where $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues of the initial matrix pair and **I** is the identity matrix.

- V. Hari: *On the global convergence of the block Jacobi method for the positive definite generalized eigenvalue problem*. Calcolo 58 (24) (2021).

J-JACOBI METHOD

PGEP for matrix pair $(\mathbf{A}, \mathbf{J})$, where $\mathbf{A}$ is symmetric positive definite and

$$\mathbf{J} = \begin{bmatrix} I_\nu & 0 \\ 0 & -I_{n-\nu} \end{bmatrix}.$$

- V. Hari, S. Singer, S. Singer: *Full block J-Jacobi method for Hermitian matrices*. Linear Algebra Appl. 444 (2014) 1–27.

Theorem (BK, Hari)

Let $\pi = (n_1, n_2, \ldots, n_m)$ be a partition of n, so that π refines $(\nu, n-\nu)$. The full block J-Jacobi method defined by the cyclic pivot ordering $\mathcal{O} \in \mathcal{B}_{sg}^{(m)}$, which uses UBCE J-orthogonal transformation matrices is globally convergent.

CHOLESKY-JACOBI (CJ) METHOD

$$\mathbf{A}x = \lambda \mathbf{B}x, \quad x \neq 0,$$

where A, B are symmetric matrices and B is positive definite.

Here we have $\pi = (1, 1, ..., 1)$, so this is element-wise method.

$$\mathbf{A}^{(0)} = \mathbf{DAD}, \quad \mathbf{B}^{(0)} = \mathbf{DBD}, \quad \mathbf{D} = \mathrm{diag}(b_{11}^{-\frac{1}{2}}, \ldots, b_{nn}^{-\frac{1}{2}})$$

$$\mathbf{A}^{(k+1)} = \mathbf{Z}_k^T \mathbf{A}^{(k)} \mathbf{Z}_k, \quad \mathbf{B}^{(k+1)} = \mathbf{Z}_k^T \mathbf{B}^{(k)} \mathbf{Z}_k, \quad k \geq 0$$

- V. Hari: *Globally convergent Jacobi methods for positive definite matrix pairs*. Numer. Algor. 79 (1) (2018) 221–249.

Theorem (BK, Hari)

CJ method is globally convergent under the class of generalized serial pivot strategies.

SOME DIFFERENT USES OF THE JACOBI-TYPE METHODS

- Structured matrix diagonalization

- EBK, H. Fassbender and P. Saltenberger: *On normal and structured matrices under unitary structure-preserving transformations*. Linear Algebra Appl. 608 (2021) 322–342.
- EBK: *Finding the closest normal structured matrix*. Linear Algebra Appl. 617 (2021) 49–77.

- Structured tensor transformations

- M. Ishteva, P.-A. Absil, P. Van Dooren: *Jacobi Algorithm for the Best Low Multilinear Rank Approximation of Symmetric Tensors*. SIAM J. Matrix Anal. Appl. 34(2) (2013) 651–672.
- EBK, D. Kressner: *Structure-preserving low multilinear rank approximation of antisymmetric tensors*. SIAM. J. Matrix Anal. Appl. 38(3) (2017) 967–983.
- J. Li, K. Usevich, P. Comon: *Globally Convergent Jacobi-Type Algorithms for Simultaneous Orthogonal Symmetric Tensor Diagonalization*. SIAM J. Matrix Anal. Appl. 39(1) (2018) 1–22.

SUMMARY

- Using reverse orderings and permutations, the class of "convergent" pivot strategies is further enlarged.
- We defined a class of Jacobi annihilators and operators designed for the block Jacobi method for symmetric matrices.
- This brings a more general view at block Jacobi method which can be used in the convergence considerations.
- The convergence results are given in a stronger form

$$S(\mathbf{A}') \leq \gamma S(\mathbf{A}), \quad 0 < \gamma \leq 1,$$

  where $\mathbf{A}'$ is the matrix obtained from $\mathbf{A}$ after one full cycle and constant γ does not depend on $\mathbf{A}$.
- This approach can be used for more general block Jacobi-type methods.

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- EBK, V. Hari: *On the global convergence of the Jacobi method for symmetric matrices of order 4 under parallel strategies*. Linear Algebra Appl. 524 (2017) 199-234.
- V. Hari, EBK: *Convergence of the cyclic and quasi-cyclic block Jacobi methods*. Electron. Trans. Numer. Anal. 46 (2017) 107-147.
- EBK, V. Hari: *Jacobi method for symmetric 4x4 matrices converges for every cyclic pivot strategy*. Numer. Algor. 78(3) (2018) 701-720.
- V. Hari, EBK: *On the convergence of complex Jacobi methods*. Linear Multilinear Algebra 69(3) (2021) 489-514.

THANK YOU!