1. KOLOKVIJ IZ MATEMATIKE I, DRUGI DIO - GRUPA B 26. studenog 2005.

1. (i) Odredite trigonometrijski prikaz brojeva $z_1 = 1 - i\sqrt{3}$, $z_2 = 1 + i\sqrt{3}$.
   (ii) Odredite z_1^3, z_2^3 i $z_1 z_2$.
   (iii) Predočite brojeve iz (ii) u kompleksnoj ravnini.

2. Zadana su tri vrha paralelograma $ABCD$ (tim redoslijedom): $A(-1,1,0)$, $B(0,1,1)$, $C(1,2,1)$.
   (i) Odredite koordinate točke D i koordinate sjecišta dijagonala paralelograma.
   (ii) Napišite matricu rotacije za $45°$ u prostoru oko osi z.
   (iii) Koristeći matricu rotacije iz (ii) zarotirajte zadani paralelogram za $45°$ oko osi z.

3. Pomoću elementarnih matričnih transformacija:
   (i) izračunajte determinantu matrice $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.
   (ii) nađite inverz matrice iz (i).
   (iii) riješite sustav
$$\begin{aligned} y - z &= -2 \\ -x + z &= 1 \\ x + y - z &= 0. \end{aligned}$$

4. Zadani su vektori $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = -\vec{i} - \vec{j} - \vec{k}$, $\vec{c} = 2\vec{i} - 2\vec{k}$.
   (i) Provjerite da su svi ti vektori međusobno ortogonalni.
   (ii) Izračunajte kut između vektora $\vec{a} + \vec{c}$ i $\vec{b}$.
   (iii) Izračunajte mješoviti produkt vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$. Jesu li ti vektori komplanarni?

5. (i) Napišite u matričnom obliku sustav
$$\begin{aligned} y - z &= -2 \\ -x + z &= 1 \\ x + y - 2z &= -3 \end{aligned}$$
   i provjerite da matrica tog sustava nije invertibilna.
   (ii) Riješite pomoću elementarnih matričnih transformacija sustav (i).
   (iii) Geometrijski interpretirajte skup rješenja.

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1\. KOLOKVIJ IZ MATEMATIKE I, DRUGI DIO – GRUPA B

1\. (i) $z_1 = 1 - i\sqrt{3}$

$|z_1| = \sqrt{1 + \sqrt{3}^2} = 2$

$\left.\begin{array}{l}\cos\alpha = \frac{1}{2}\\ \sin\alpha = -\frac{\sqrt{3}}{2}\end{array}\right\} \Rightarrow \alpha = 300^\circ$

$z_1 = 2(\cos 300^\circ + i\sin 300^\circ)$

$z_2 = 1 + i\sqrt{3}$

$|z_2| = 2$

$\left.\begin{array}{l}\cos\alpha = \frac{1}{2}\\ \sin\alpha = \frac{\sqrt{3}}{2}\end{array}\right\} \Rightarrow \alpha = 60^\circ$

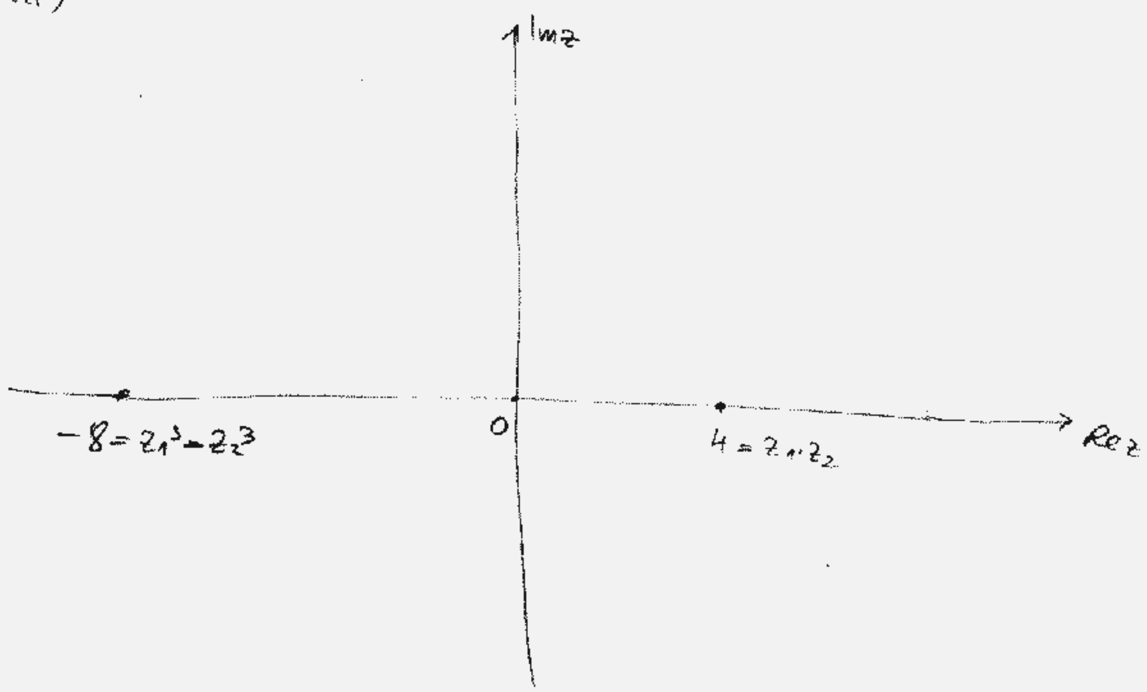
$z_2 = 2(\cos 60^\circ + i\sin 60^\circ)$

(ii) $z_1^3 = 2^3(\cos 3\cdot 300^\circ + i\sin 3\cdot 300^\circ) = 8(\cos 900^\circ + i\sin 900^\circ) = \ldots = 8(\underbrace{\cos 180^\circ}_{-1} + i\underbrace{\sin 180^\circ}_{0}) = -8$

$z_2^3 = 2^3(\cos 3\cdot 60^\circ + i\sin 3\cdot 60^\circ) = 8(\underbrace{\cos 180^\circ}_{-1} + i\underbrace{\sin 180^\circ}_{0}) = -8$

$z_1 \cdot z_2 = 2\cdot 2(\cos(60^\circ + 300^\circ) + i\sin(60^\circ + 300^\circ)) = 4(\underbrace{\cos 360^\circ}_{1} + i\underbrace{\sin 360^\circ}_{0}) = 4$

(iii)



2. (i)

$\vec{AD} = \vec{BC}$, $D = (x,y,z)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \boxed{D(0,2,0)}$$

$S = (x,y,z)$ $\quad \vec{AS} = \frac{1}{2}\vec{AC}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 1/2 \end{bmatrix} \Rightarrow \boxed{S\left(-1, \frac{3}{2}, \frac{1}{2}\right)}$$

(ii) $\alpha = 45°$

$$R = \begin{bmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) $ABCD \xrightarrow{R} A'B'C'D'$

$$A': \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad A'(-\sqrt{2}, 0, 0)$$

$$B': \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} \Rightarrow \quad B'\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$$

$$C': \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} - \sqrt{2} \\ \frac{\sqrt{2}}{2} + \sqrt{2} \\ 1 \end{bmatrix} \Rightarrow C'\left(-\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right)$$

$$D': \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \Rightarrow \quad D'(-\sqrt{2}, \sqrt{2}, 0)$$

3. (i)

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \underset{\substack{\mathrm{I} \to \mathrm{I} - \mathrm{III} \\ \mathrm{II} \to \mathrm{II} + \mathrm{III}}}{=} \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -1 \cdot 1 \cdot (-1) = 1$$

(ii)

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\mathrm{I} \leftrightarrow \mathrm{III}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \underset{\mathrm{II} \to \mathrm{II} + \mathrm{I}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \underset{\substack{\mathrm{I} \to \mathrm{I} - \mathrm{II} \\ \mathrm{III} \to \mathrm{III} - \mathrm{II}}}{\sim}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & -1 \end{array}\right] / \cdot (-1) = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \underset{\mathrm{I} \to \mathrm{I} + \mathrm{III}}{\sim}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \Rightarrow \text{inverz je } \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

(iii) Radimo iste transformacije kao gore:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 3 \end{aligned}$$

4. (i) $\vec{a}\cdot\vec{b} = 1\cdot(-1) - 2\cdot(-1) + 1\cdot(-1) = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{a}\cdot\vec{c} = 1\cdot 2 + (-2)\cdot 0 + 1\cdot(-2) = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\vec{b}\cdot\vec{c} = -1\cdot 2 - 1\cdot 0 - 1\cdot(-2) = 0 \Rightarrow \vec{b} \perp \vec{c}$

(ii) $\vec{a} + \vec{c} = 3\vec{i} - 2\vec{j} - \vec{k}$

$(\vec{a}+\vec{c})\cdot\vec{b} = 3\cdot(-1) + (-2)\cdot(-1) + (-1)\cdot(-1) = -3 + 2 + 1 = 0$

Logično, $\vec{a}+\vec{c} \perp \vec{b}$ (jer $\vec{a} \perp \vec{b}$ i $\vec{c} \perp \vec{b}$) pa je kut među zadanim vektorima $90°$.

(iii) $$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 1^{+} & -2 & 1 \\ -1^{-} & -1 & -1 \\ 2^{+} & 0^{-} & -2^{+} \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} =$$

$$= 2\cdot(2+1) - 2\cdot(-1-2) = 2\cdot 3 - 2\cdot(-3) = 6 + 6 = 12 \neq 0$$

$\Rightarrow$ vektori nisu komplanarni

5. (i) $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ matrica sustava

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ matrica nepoznanica

$B = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ matrica slobodnih koeficijenata

matrični zapis sustava $A \cdot X = B$, uz A, X i B kao gore.

Da provjerimo da matrica sustava nije invertibilna dovoljno je vidjeti da je $\det A = 0$:

$$\begin{vmatrix} \overset{+}{0} & \overset{-}{1} & \overset{+}{-1} \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -(2-1) - (-1) = -1+1 = 0$$

(ii)
$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -2 & -3 \end{array}\right] \underset{\mathrm{I} \leftrightarrow \mathrm{III}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right] \underset{\mathrm{II} \to \mathrm{II} + \mathrm{I}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ \boxed{0 \;\; 1 \;\; -1 \;\; -2} \end{array}\right] \sim$$

isto kao 2. redak ⇒ možemo ga izbaciti

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -1 & -2 \end{array}\right] \underset{\mathrm{I} \to \mathrm{I} - \mathrm{II}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$x - z = -1$$
$$y - z = -2$$

Uzmimo $z = t, t \in \mathbb{R} \Rightarrow \begin{array}{l} x = t-1 \\ y = t-2 \end{array}$

⇒ Rješenje je $\begin{cases} x = t-1 \\ y = t-2 \\ z = t \\ t \in \mathbb{R} \end{cases}$ ⇒ sustav ima beskonačno mnogo rješenja

(iii) Geometrijska interpretacija: to je pravac

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{vektor smjera}} + \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}}_{\text{točka kroz koju pravac prolazi}}$$