3. KOLOKVIJ IZ MATEMATIKE 1, 2.010 - GRUPA A 1.2.2006.

1.(i) Izračunajte $\lim\limits_{x \to 3} \dfrac{x^2-6x+9}{x^2-9x+18}$

(ii) Izračunajte $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{3x^2}$.

(iii) Derivirajte funkciju $f(x) = \ln\sqrt{2-x^2} + \sqrt{1-\ln^3 x}$.

(iv) Odredite tangentu na graf funkcije $f(x) = \dfrac{x^2+1}{x^3+2}$ u točki $(1, f(1))$.

Rješenje. (i) $\lim\limits_{x \to 3} \dfrac{x^2-6x+9}{x^2-9x+18} = \left(\dfrac{0}{0}\right) = \lim\limits_{x \to 3} \dfrac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}(x-6)} = \dfrac{0}{-3} = 0$

(ii) $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{3x^2} = \lim\limits_{x \to 0} \dfrac{1}{3} \cdot \underbrace{\dfrac{\sin x}{x}}_{\to 1} \cdot \underbrace{\dfrac{\sin x}{x}}_{\to 1} = \dfrac{1}{3}$

(iii) $f'(x) = \dfrac{1}{\sqrt{2-x^2}} \cdot \dfrac{1}{2\sqrt{2-x^2}} \cdot (-2x) + \dfrac{1}{2\sqrt{1-\ln^3 x}} \cdot (-3\ln^2 x) \cdot \dfrac{1}{x} =$

$= \dfrac{-x}{2-x^2} - \dfrac{3\ln^2 x}{2x\sqrt{1-\ln^3 x}}$

(iv)

$y - f(x_0) = f'(x_0)\cdot(x - x_0)$ jednadžba tangente

$x_0 = 1$

$f(x_0) = \dfrac{1+1}{1+2} = \dfrac{2}{3}$

$f'(x) = \dfrac{2x\cdot(x^3+2) - (x^2+1)\cdot 3x^2}{(x^3+2)^2} = \dfrac{2x^4+4x-3x^4-3x^2}{(x^3+2)^2} =$

$= \dfrac{-x^4-3x^2+4x}{(x^3+2)^2}$

$f'(1) = \dfrac{-1-3+4}{(1+2)^2} = 0$

$\Rightarrow$ tangenta je $y - \dfrac{2}{3} = 0\cdot(x-1) \Rightarrow \boxed{y = \dfrac{2}{3}}$

2. (i) Koristeći linearnu aproksimaciju izračunajte približno $\sqrt[3]{8.02^2}$

(ii) Koristeći kvadratnu aproksimaciju izračunajte približno $\sqrt[3]{8.02^2}$

<u>Rješenje.</u> (i) $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x$

$x_0 = 8$, $\Delta x = 0.02$ $\quad f(x) = \sqrt[3]{x^2}$

$$\Rightarrow f(x_0) = f(8) = \sqrt[3]{8^2} = 2^2 = 4$$

$$f'(x) = (x^{2/3})' = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

$$f'(x_0) = f'(8) = \frac{2}{3\sqrt[3]{8}} = \frac{\cancel{2}}{3\cdot\cancel{2}} = \frac{1}{3}$$

$$\sqrt[3]{8.02^2} = f(8+0.02) \approx 4 + \frac{1}{3}\cdot 0.02 = 4 + \frac{1}{3}\cdot\frac{\cancel{2}}{\cancel{100}_{50}} = 4 + \frac{1}{150}$$

$$\boxed{\sqrt[3]{8.02^2} \approx 4 + \frac{1}{150}}$$

(ii) $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x + \frac{1}{2}f''(x_0)(\Delta x)^2$

$$f''(x) = \left(\frac{2}{3}x^{-1/3}\right)' = \frac{2}{3}\cdot\frac{-1}{3}x^{-4/3} = \frac{-2}{9\sqrt[3]{x^4}}$$

$$\Rightarrow f''(x_0) = f''(8) = \frac{-2}{9\sqrt[3]{8^4}} = \frac{-\cancel{2}}{9\cdot 2^{\cancel{4}}} = \frac{-1}{9\cdot 8} = \frac{-1}{72}$$

$$\Rightarrow \sqrt[3]{8.02^2} = f(8+0.02) \approx 4 + \frac{1}{150} + \frac{1}{2}\cdot\frac{-1}{72}\cdot\left(\frac{\cancel{2}}{\cancel{100}_{50}}\right)^2 =$$

$$= 4 + \frac{1}{150} - \frac{1}{2\cdot 72}\cdot\frac{1}{2500} =$$

$$= 4 + \frac{1}{150} - \frac{1}{360000}$$

$$\Rightarrow \boxed{\sqrt[3]{8.02^2} \approx 4 + \frac{1}{150} - \frac{1}{360000}}$$

(i) Razvijte u Taylorov red oko $x_0=0$ funkciju $f(x)=\frac{3}{1-2x}$.

(ii) Napišite prvih 8 članova dobivenog Taylorovog reda.

(iii) Izračunajte $f^{(100)}(0)$.

(iv) Nađite područje konvergencije tog reda.

Rješenje (i) $f(x)=\frac{3}{1-2x}=3\cdot\frac{1}{1-(2x)} \Rightarrow$

Taylorov red je $T(x)=3\cdot\sum_{n=0}^{\infty}(2x)^n=\sum_{n=0}^{\infty}3\cdot 2^n x^n$

(ii)

$$\sum_{n=0}^{\infty}3\cdot 2^n\cdot x^n = 3+3\cdot 2x+3\cdot 2^2 x^2+3\cdot 2^3 x^3+3\cdot 2^4 x^4+$$
$$+3\cdot 2^5 x^5+3\cdot 2^6 x^6+3\cdot 2^7 x^7+\dots$$

(iii) Iz formule za Taylorov red oko nule

$$T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Vidimo da je $\frac{f^{(100)}(0)}{100!}$ koeficijent uz x^{100}, pa mora biti

$$\frac{f^{(100)}(0)}{100!}=3\cdot 2^{100} \Rightarrow \boxed{f^{(100)}(0)=3\cdot 2^{100}\cdot 100!}$$

(iv)

Geom. red $\sum x^n$ konvergira za $|x|<1$, pa

naš Taylorov red $3\sum(2x)^n$ konvergira za $|2x|<1$, tj.

$$2|x|<1$$
$$|x|<1/2$$

$\rightarrow$ $\boxed{x\in\langle-\frac{1}{2},\frac{1}{2}\rangle}$ je područje konvergencije

4.15 Zadana je funkcija $f(x) = \frac{3x^2+12}{x}$. Odredite:

(i) domenu funkcije

(ii) njene nultočke

(iii) asimptote (horizontalne, kose i vertikalne),

(iv) lokalne ekstreme,

(v) područja pada i rasta

(vi) područja konveksnosti, konkavnosti i točke infleksije.

(vii) Nacrtajte precizno graf te funkcije koristeći gornje podatke.

Rješenje

(i) $D(f) = \mathbb{R} \setminus \{0\}$ (zbog nazivnika)

(ii) $N(f) = \emptyset$, jer $3x^2+12 \neq 0$ za sve $x \in \mathbb{R}$

(iii) asimptote:

* horizontalna $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2+12}{x} \;/:x^2 = \lim_{x\to\infty} \frac{3+\frac{12}{x^2}\to 0}{\frac{1}{x}\to 0} = \infty$ nema

* kosa $y = kx + l$

$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3x^2+12}{x^2} \;/:x^2 = \lim_{x\to\infty} \frac{3+\frac{12}{x^2}\to 0}{1} = 3$

$l = \lim_{x\to\infty} (f(x) - 3x) = \lim_{x\to\infty} \left(\frac{3x^2+12}{x} - 3x \right) =$

$= \lim_{x\to\infty} \left(\frac{\cancel{3x^2}+12-\cancel{3x^2}}{x} \right) = \lim_{x\to\infty} \frac{12}{x}\to 0 = 0$

$\Rightarrow \quad y = 3x$

* vertikalna $x = 0$

(iv)

$f'(x) = \frac{6x\,x - (3x^2+12)\cdot 1}{x^2} = \frac{3x^2-12}{x^2} = 0$

$3x^2 = 12$

$x^2 = 4$

$x_{1,2} = \pm 2$ kritične točke

$f''(x) = \frac{6x\cdot x^{\cancel{2}} - (3x^2-12)\cdot 2\cancel{x}}{x^{\cancel{4}3}} = \frac{\cancel{6x^3}-\cancel{6x^3}+24}{x^3} = \frac{24}{x^3}$

$f''(2) = \frac{24}{8} = 3 > 0 \Rightarrow (2, f(2))$ lok. min.

$f(2) = \frac{3\cdot 4+12}{2} = 12 \Rightarrow \boxed{(2,12) \text{ lok. min}}$

$f''(-2) = \frac{24}{-8} = -3 < 0 \Rightarrow (-2, f(-2))$ lok. max.

$f(-2) = \frac{3\cdot 4 + 12}{-2} = -12 \rightarrow \boxed{(-2,-12) \text{ lok. maks.}}$

(v) rast: $f'(x) > 0 \Rightarrow \frac{3x^2-12}{x^2} > 0$

$x^2 > 0 \quad \forall x \in \mathbb{R}\setminus\{0\}$

$\Rightarrow 3x^2 - 12 > 0$

$3x^2 > 12$

$x^2 > 4 \Rightarrow |x| > 2 \Rightarrow \boxed{x \in \langle -\infty, -2\rangle \cup \langle 2, \infty\rangle \text{ rast}}$

$\Rightarrow \boxed{x \in \langle -2, 2\rangle \setminus \{0\} \text{ pad}}$

(vi) konveksnost: $f''(x) > 0 \Rightarrow \frac{24}{x^3} > 0 \Rightarrow x > 0$

$x^3 = x^2 \cdot x$, $x^2 > 0 \quad \forall x \in \mathbb{R}$

$\boxed{x \in \langle 0, \infty\rangle \text{ konveksnost}}$

$\Downarrow$

$\boxed{x \in \langle -\infty, 0\rangle \text{ konkavnost}}$

točka infleksije: $f''(x) = 0 \Rightarrow \frac{24}{x^3} = 0$ nema; ali $x = 0$ (koja nije u $D(f)$) je dodir $\Rightarrow \boxed{x = 0 \text{ točka infleksije}}$

(vii)