Poglavlje 1

Neodređeni integral i metode računanja

1.1 Primitivna funkcija

Postupak dobivanja neodređenog integrala obrnut je postupku dobivanja derivacije funkcije. Ako je zadana funkcija f, pitamo se što je potrebno derivirati kako bi se upravo ona dobila kao rezultat deriviranja.

Definicija 1.1.1. Za $f : \langle a, b\rangle \to \mathbb{R}$ funkciju $F : \langle a, b\rangle \to \mathbb{R}$ takvu da vrijedi $F'(x) = f(x)$, za svaki $x \in \langle a, b\rangle$, zovemo **primitivna funkcija**.

Primjer 1.1.2. Primitivne funkcije od $f(x) = 2x$ su $F(x) = x^2$, $F(x) = x^2 + 5$, $F(x) = x^2 + \pi$ itd.
Prema tome, zaključujemo da je primitivna funkcija funkcije $f(x)$ bilo koja funkcija oblika $F(x) = x^2 + C, C \in \mathbb{R}$.

Definicija 1.1.3. Skup svih primitivnih funkcija od f nazivamo **neodređeni integral** te označavamo s $\int f(x)\,dx := \{F(x) | F'(x) = f(x), \text{ za svaki } x \in \langle a, b\rangle\}$.

Zadatak 1.1.4. Odredite sve primitivne funkcije od $f(x)$ ako je

a) $f(x) = x^3$, b) $f(x) = \frac{1}{\sqrt{x}}$, c) $f(x) = e^x$, d) $f(x) = x$, e) $f(x) = \frac{1}{x}$.

Rješenje.
Pitamo se koju funkciju trebamo derivirati da bismo dobili funkciju $f(x)$.

a) $F(x) = \frac{1}{4}x^4 + C$, $C \in \mathbb{R}$

b) $f(x) = x^{-\frac{1}{2}} \Rightarrow F(x) = 2x^{\frac{1}{2}} + C$, $C \in \mathbb{R}$

c) $F(x) = e^x + C,\ C \in \mathbb{R}$

d) $F(x) = \frac{1}{2}x^2 + C,\ C \in \mathbb{R}$

e) $F(x) = \ln x + C,\ C \in \mathbb{R}$

Kako ne bismo ubuduće morali tražiti skup primitivnih funkcija na način na koji smo to radili u prošlom zadatku, zapisujemo tablicu integrala koja sadrži integrale elementarnih funkcija. Dostupna je na linku.

Teorem 1.1.5. Svojstva integrala

a) $\displaystyle\int (f(x) + g(x))\ dx = \int f(x)\,dx + \int g(x)\,dx,$ **aditivnost(A)**

b) $\displaystyle\int \alpha f(x)\,dx = \alpha \int f(x)\,dx, \text{za } \alpha \in \mathbb{R},$ **homogenost(H)**

c) $\displaystyle\int f'(x)\,dx = f(x) + C,\ C \in \mathbb{R}$

Zadatak 1.1.6. Odredite neodređene integrale:

a) $\displaystyle\int \left(x^2 + x + 1\right)\ dx,$ b) $\displaystyle\int \left(\sqrt{x} + \cos x\right)\ dx,$ c) $\displaystyle\int (1 + 2\cos x)\ dx,$
d) $\displaystyle\int \left(2x^2 + 1\right)^3\ dx.$

Rješenje.

a)

$$\int \left(x^2 + x + 1\right)\ dx \overset{A}{=} \int x^2\,dx + \int x\,dx + \int 1\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C,\ C \in \mathbb{R}$$

b)

$$\int \left(\sqrt{x} + \cos x\right)\ dx \overset{A}{=} \int \sqrt{x}\,dx + \int \cos x\,dx = \int x^{\frac{1}{2}}\,dx + \int \cos x\,dx =$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \sin x + C = \frac{2}{3}\sqrt{x}^3 + \sin x + C,\ C \in \mathbb{R}$$

c)

$$\int (1 + 2\cos x)\ dx \overset{A}{=} \int 1\,dx + \int 2\cos x\,dx \overset{H}{=} \int 1\,dx + 2\int \cos x\,dx =$$
$$= x + 2\sin x + C,\ C \in \mathbb{R}$$

d)

$$\int \left(2x^2+1\right)^3\,dx = \int \left(8x^6+12x^4+6x^2+1\right)\,dx =$$
$$\stackrel{AiH}{=} 8\int x^6\,dx + 12\int x^4\,dx + 6\int x^2\,dx + \int 1dx =$$
$$= \frac{8}{7}x^7 + \frac{12}{5}x^5 + 2x^3 + x + C,\; C \in \mathbb{R}$$

Primjer 1.1.7. $\int \frac{1}{x}\,dx = \ln|x| + C,\; C \in \mathbb{R}$

Zašto?

Za $x > 0$ imamo $(\ln x + C)' = \frac{1}{x}$.

Za $x < 0$ imamo $(\ln(-x) + C)' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$.

1.2 Metode rješavanja neodređenih integrala

- metoda supstitucije

- metoda parcijalne integracije

1.2.1 Metoda supstitucije

Primjer 1.2.1. Odrediti sljedeće integrale:

a) $\int \frac{1}{x+2}\,dx$, b) $\int \frac{1}{2x+1}\,dx$.

Kod integrala pod a) naslućujemo da bi se moglo raditi o $\ln(x+2)$. Ako deriviramo $\ln(x+2)$, vidimo da dobijemo isto što piše pod znakom integrala. Uzmemo li u obzir prethodni primjer, rješenje je jednako $\ln|x+2| + C,\; C \in \mathbb{R}$. Kod integrala pod b), vodimo li se istom idejom, naslućujemo da je u pitanju $\ln(2x+1)$. No, ovaj se put u derivaciji izraza $\ln(2x+1)$ javlja konstanta 2 koje nema kod podintegralne funkcije. Kako bi se ta konstanta poništila, primitivna funkcija mora sadržavati 1/2, tj. rješenje je $\frac{1}{2}\ln|2x+1| + C,\; C \in \mathbb{R}$. Kako ne bismo kod zadataka ovakvog tipa "pogađali" rješenje, koristimo metodu supstitucije.

Zadatak 1.2.2. Odredite integrale:

a) $\int \frac{1}{x+2}\,dx$, b) $\int \frac{1}{2x+1}\,dx$.

Rješenje.

a)

$$\int \frac{1}{x+2}\,dx = \int \frac{1}{x+2}\,dx = \begin{bmatrix} t = x+2 \\ dt = dx \end{bmatrix} = \int \frac{1}{t}\,dt = \ln|t| + C =$$

$$= \ln|x+2| + C,\ C \in \mathbb{R}$$

b)

$$\int \frac{1}{2x+1}\,dx = \int \frac{1}{2x+1}\,dx = \begin{bmatrix} t = 2x+1 \\ dt = 2dx \\ dx = \frac{dt}{2} \end{bmatrix} = \int \frac{1}{t}\cdot\frac{1}{2}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt =$$

$$= \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|2x+1| + C,\ C \in \mathbb{R}$$

Teorem 1.2.3. Formula supstitucije u integralu

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \begin{bmatrix} t = \varphi(x) \\ dt = \varphi'(x)\,dx \end{bmatrix} = \int f(t)\,dt$$

Zadatak 1.2.4. Riješite metodom supstitucije:

a) $\displaystyle\int \frac{e^x}{e^x+1}\,dx,$ b) $\displaystyle\int \frac{\ln x}{x}\,dx,$ c) $\displaystyle\int \sin^3 x \cos x\,dx.$

Rješenje.

a)

$$\int \frac{e^x}{e^x+1}\,dx = \begin{bmatrix} t = e^x+1 \\ dt = e^x\,dx \end{bmatrix} = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|e^x+1| + C =$$

$$= \ln(e^x+1) + C,\ C \in \mathbb{R}$$

b)

$$\int \frac{\ln x}{x}\,dx = \begin{bmatrix} t = \ln x \\ dt = \frac{1}{x}\,dx \end{bmatrix} = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}\ln^2 x + C,\ C \in \mathbb{R}$$

c)

$$\int \sin^3 x \cos x\,dx = \begin{bmatrix} t = \sin x \\ dt = \cos x\,dx \end{bmatrix} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{1}{4}\sin^4 x + C, C \in \mathbb{R}$$

Zadatak 1.2.5. Riješite metodom supstitucije:

a) $\int \left(2x^2+1\right)^{46} x\,dx$, b) $\int x\sqrt{2-5x}\,dx$, c) $\int \frac{x}{\sqrt[3]{1-3x}}\,dx$, d) $\int x\sqrt{2-5x^2}\,dx$.

Rješenje.

a)

$$\int \left(2x^2+1\right)^{46} x\,dx = \begin{bmatrix} t=2x^2+1 \\ dt = 4x\,dx \\ dx = \frac{1}{4}\,dt \end{bmatrix} = \int t^{46}\,\frac{1}{4}\,dt = \frac{1}{4}\int t^{46}\,dt = \frac{1}{4}\frac{t^{47}}{47} + C =$$
$$= \frac{1}{188}(2x^2+1)^{47} + C,\ C\in\mathbb{R}$$

b)

$$\int x\sqrt{2-5x}\,dx = \begin{bmatrix} t=2-5x \Rightarrow x = \frac{-t+2}{5} \\ dt = -5\,dx \\ dx = -\frac{1}{5}\,dt \end{bmatrix} = \int \frac{-t+2}{5}\sqrt{t}\cdot\left(-\frac{1}{5}\,dt\right) = \frac{1}{25}\int (t-2)\sqrt{t}\,dt =$$
$$= \frac{1}{25}\int\left(t\sqrt{t}-2\sqrt{t}\right)\,dt = \frac{1}{25}\int\left(t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\,dt =$$
$$= \frac{1}{25}\left(\int t^{\frac{3}{2}}\,dt - 2\int t^{\frac{1}{2}}\,dt\right) = \frac{1}{25}\left(\frac{2}{5}t^{\frac{2}{5}} - 2\cdot\frac{2}{3}t^{\frac{3}{2}}\right) + C =$$
$$= \frac{1}{25}\left(\frac{2}{5}\sqrt{t}^5 - \frac{4}{3}\sqrt{t}^3\right) + C = \frac{1}{25}\left(\frac{2}{5}\sqrt{2-5x}^5 - \frac{4}{3}\sqrt{2-5x}^3\right) + C,\ C\in\mathbb{R}$$

c)

$$\int \frac{x}{\sqrt[3]{1-3x}}\,dx = \begin{bmatrix} t=1-3x \Rightarrow x = \frac{1-t}{3} \\ dt = -3\,dx \\ dx = -\frac{1}{3}\,dt \end{bmatrix} = \int \frac{\frac{1-t}{3}}{\sqrt[3]{t}}\cdot\left(-\frac{1}{3}\,dt\right) = \int \frac{1-t}{3t^{\frac{1}{3}}}\cdot\left(-\frac{1}{3}\,dt\right) =$$
$$= -\frac{1}{9}\int\frac{1-t}{t^{\frac{1}{3}}}\,dt = -\frac{1}{9}\left(\int t^{-\frac{1}{3}}\,dt - \int\frac{t}{t^{\frac{1}{3}}}\right) = -\frac{1}{9}\left(\int t^{-\frac{1}{3}}\,dt - \int t^{\frac{2}{3}}\,dt\right) =$$
$$= -\frac{1}{9}\left(\frac{3}{2}t^{\frac{2}{3}} - \frac{3}{5}t^{\frac{5}{3}}\right) + C = -\frac{1}{9}\left(\frac{3}{2}\sqrt[3]{(1-3x)^2} - \frac{3}{5}\sqrt[3]{(1-3x)^5}\right) + C =$$
$$= -\frac{1}{6}\sqrt[3]{(1-3x)^2} + \frac{1}{15}\sqrt[3]{(1-3x)^5} + C,\ C\in\mathbb{R}$$

d)

$$\int x\sqrt{2-5x^2}\,dx = \begin{bmatrix} t=2-5x^2 \\ dt = -10x\,dx \\ x\,dx = -\frac{1}{10}\,dt \end{bmatrix} = \int\sqrt{t}\cdot\left(-\frac{1}{10}\,dt\right) = -\frac{1}{10}\int t^{\frac{1}{2}}\,dt = -\frac{1}{10}\cdot\frac{2}{3}t^{\frac{3}{2}} + C =$$
$$= -\frac{1}{15}\sqrt{2-5x^2}^3 + C,\ C\in\mathbb{R}$$

Zadatak 1.2.6. Riješite metodom supstitucije:

a) $\int e^{3\cos x}\sin x\,dx$, b) $\int \frac{2^{\operatorname{arctg} x}}{1+x^2}\,dx$, c) $\int x^2 e^{x^3}\,dx$.

Rješenje.

a)

$$\int e^{3\cos x}\sin x\,dx = \begin{bmatrix} t = 3\cos x \\ dt = -3\sin x\,dx \\ \sin x\,dx = -\frac{1}{3}\,dt \end{bmatrix} = \int e^t\cdot\left(-\frac{1}{3}\,dt\right) = -\frac{1}{3}\int e^t\,dt = -\frac{1}{3}e^t + C =$$
$$= -\frac{1}{3}e^{3\cos x} + C,\ C\in\mathbb{R}$$

b)

$$\int \frac{2^{\operatorname{arctg} x}}{1+x^2}\,dx = \begin{bmatrix} t = \operatorname{arctg} x \\ dt = \frac{1}{1+x^2}\,dx \end{bmatrix} = \int 2^t\,dt = \frac{2^t}{\ln 2} + C = \frac{1}{\ln 2}2^{\operatorname{arctg} x} + C,\ C\in\mathbb{R}$$

c)

$$\int x^2 e^{x^3}\,dx = \begin{bmatrix} t = x^3 \\ dt = 3x^2\,dx \\ x^2\,dx = \frac{1}{3}\,dt \end{bmatrix} = \int e^t\cdot\frac{1}{3}\,dt = \frac{1}{3}\int e^t\,dt = \frac{1}{3}e^t + C = \frac{1}{3}e^{x^3} + C,\ C\in\mathbb{R}$$

Zadatak 1.2.7. Riješite integrale:

a) $\int 3^x e^x\,dx$, b) $\int \left(1+\sqrt{x}\right)^4\,dx$, c) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}}$.

Rješenje.

a)

$$\int 3^x e^x\,dx = \int (3e)^x\,dx = \frac{(3e)^x}{\ln(3e)} + C,\ C\in\mathbb{R}$$

b)

$$\int \left(1+\sqrt{x}\right)^4\,dx = \begin{bmatrix} t = 1+\sqrt{x} \Rightarrow \sqrt{x} = t-1 \\ dt = \frac{1}{2}x^{-\frac{1}{2}}\,dx \\ dt = \frac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2(t-1)\,dt \end{bmatrix} = \int t^4\cdot 2(t-1)\,dt =$$
$$= 2\int \left(t^5 - t^4\right)\,dt = 2\left(\int t^5\,dt - \int t^4\,dt\right) = \frac{1}{3}t^6 - \frac{2}{5}t^5 + C =$$
$$= \frac{1}{3}\left(1+\sqrt{x}\right)^6 - \frac{2}{5}\left(1+\sqrt{x}\right)^5 + C,\ C\in\mathbb{R}$$

c)

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} = \int \frac{1}{\sqrt{x+1}+\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\,dx =$$
$$= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{(x+1)-(x-1)}\,dx = \int \frac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx =$$
$$= \int \frac{1}{2}\left(\int \sqrt{x+1}\,dx - \int \sqrt{x-1}\,dx\right) =$$
$$= \begin{bmatrix} t = x+1 & s = x-1 \\ dt = dx & ds = dx \end{bmatrix} = \frac{1}{2}\left(\int t^{\frac{1}{2}}\,dt - \int s^{\frac{1}{2}}\,ds\right) =$$
$$= \frac{1}{2}\left(\frac{2}{3}t^{\frac{3}{2}} - \frac{2}{3}s^{\frac{3}{2}}\right) + C = \frac{1}{3}\sqrt{x+1}^3 - \frac{1}{3}\sqrt{x-1}^3 + C,\ C \in \mathbb{R}$$

Zadatak 1.2.8. Riješite integrale:

a) $\int \operatorname{tg} x\,dx$, b) $\int \operatorname{tg}^2 x\,dx$, c) $\int \frac{4}{\sin^2(2x)}\,dx$, d) $\int \frac{dx}{\sin^2 x \cos^2 x}$, e)* $\int \frac{1}{\sin x}\,dx$.

Rješenje.

a)

$$\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x\,dx \\ \sin x\,dx = -dt \end{bmatrix} = \int -\frac{dt}{t} = -\int \frac{1}{t}\,dt =$$
$$= -\ln|t| + C = -\ln|\cos x| + C,\ C \in \mathbb{R}$$

b)

$$\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right)\,dx =$$
$$= \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx = \operatorname{tg} x - x + C,\ C \in \mathbb{R}$$

c)

$$\int \frac{4}{\sin^2(2x)}\,dx = \begin{bmatrix} t = 2x \\ dt = 2\,dx \\ dx = \frac{1}{2}\,dt \end{bmatrix} = 4\int \frac{1}{\sin^2 t} \cdot \frac{1}{2}\,dt = 2\int \frac{1}{\sin^2 t}\,dt = 2 \cdot (-\operatorname{ctg} t) + C =$$
$$= -2\operatorname{ctg}(2x) + C,\ C \in \mathbb{R}$$

d) 1. način

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{4\,dx}{4\sin^2 x \cos^2 x} = \int \frac{4\,dx}{(2\sin x \cos x)^2} =$$
$$= \int \frac{4\,dx}{(\sin(2x))^2} \overset{c)}{=} -2\operatorname{ctg}(2x) + C,\ C \in \mathbb{R}$$

2. način

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{1}{\sin^2 x \cos^2 x}\, dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx =$$
$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx + \int \frac{1}{\sin^2 x}\, dx =$$
$$= \operatorname{tg} x - \operatorname{ctg} x + C,\ C \in \mathbb{R}$$

e)*

$$\int \frac{1}{\sin x}\, dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \cdot \frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}\, dx = \int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}\, dx =$$
$$= \int \frac{1}{2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}}\, dx = \frac{1}{2}\int \frac{1}{\operatorname{tg}\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}}\, dx = \begin{bmatrix} t = \operatorname{tg}\frac{x}{2} \\ dt = \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2}\, dx \\ \frac{1}{\cos^2\frac{x}{2}}\, dx = 2\, dt \end{bmatrix} =$$
$$= \frac{1}{2}\int \frac{1}{t} \cdot 2\, dt = \int \frac{1}{t}\, dt = \ln|t| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C,\ C \in \mathbb{R}$$

Zadatak 1.2.9. Riješite: $\displaystyle\int \frac{x^2+1}{x-1}\, dx.$

Rješenje. 1.način

Primjećujemo da je stupanj brojnika veći od stupnja nazivnika pa podijelimo najprije polinome.

$$\begin{array}{rl} & (x^2+1) \div (x-1) = x+1 \\ - & x^2 \quad - x \\ \hline & x+1 \\ - & x-1 \\ \hline & 2 \end{array}$$

Prema tome,

$$\frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}.$$

$$\int \frac{x^2+1}{x-1}\, dx = \int \left(x + 1 + \frac{2}{x-1}\right) dx = \int x\, dx + \int 1\, dx + 2\int \frac{1}{x-1}\, dx =$$
$$= \frac{x^2}{2} + x + 2\ln|x-1| + C,\ C \in \mathbb{R}$$

2. način

$$\int \frac{x^2+1}{x-1}\,dx = \begin{bmatrix} t = x+1 \Rightarrow x = t+1 \\ dt = dx \end{bmatrix} = \int \frac{(t+1)^2+1}{t}\,dt = \int \frac{t^2+2t+2}{t}\,dt =$$
$$= \int \left(t+2+\frac{2}{t}\right)dt = \frac{t^2}{2} + 2t + 2\ln|t| + C =$$
$$= \frac{(x-1)^2}{2} + 2(x-1) + 2\ln|x-1| + C,\ C \in \mathbb{R}$$

Napomena 1.2.10. Naizgled rješenja nisu jednaka, no ako raspišemo neko od njih (ili oba) vidjet ćemo da se radi o istom skupu primitivnih funkcija.

Zadatak 1.2.11. Riješite sljedeće integrale:

a) $\displaystyle\int \frac{dx}{\arcsin^2 x\sqrt{1-x^2}}$, b) $\displaystyle\int \frac{dx}{3^x+2}$, c) $\displaystyle\int \frac{\sin x\cos x}{\sqrt{2\sin^2 x+\cos^2 x}}\,dx$.

Rješenje.

a)

$$\int \frac{dx}{\arcsin^2 x \cdot \sqrt{1-x^2}} = \begin{bmatrix} t = \arcsin x \\ dt = \frac{1}{\sqrt{1-x^2}}\,dx \end{bmatrix} = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C =$$
$$= -\frac{1}{t} + C = -\frac{1}{\arcsin x} + C,\ C \in \mathbb{R}$$

b)

$$\int \frac{dx}{3^x+2} = \int \frac{1}{2}\cdot\frac{2}{3^x+2}\,dx = \frac{1}{2}\int \frac{3^x+2-3^x}{3^x+2}\,dx = \frac{1}{2}\int \left(\frac{3^x+2}{3^x+2} - \frac{3^x}{3^x+2}\right)dx =$$
$$= \begin{bmatrix} t = 3^x+2 \\ dt = 3^x\ln 2\,dx \\ 3^x\,dx = \frac{1}{\ln 3}\,dt \end{bmatrix} = \frac{1}{2}\left(\int 1\,dx - \int \frac{1}{t}\cdot\frac{1}{\ln 3}\,dt\right) =$$
$$= \frac{1}{2}\left(x - \frac{1}{\ln 3}\ln|t|\right) + C = \frac{1}{2}\left(x - \frac{1}{\ln 3}\ln|3^x+2|\right) + C =$$
$$= \frac{1}{2}\left(x - \frac{1}{\ln 3}\ln(3^x+2)\right) + C,\ C \in \mathbb{R}$$

c)

$$\int \frac{\sin x\cos x}{\sqrt{2\sin^2 x+\cos^2 x}}\,dx = \int \frac{\sin x\cos x}{\sqrt{\sin^2 x+\sin^2 x+\cos^2 x}}\,dx = \int \frac{\sin x\cos x}{\sqrt{\sin^2 x+1}}\,dx =$$
$$= \begin{bmatrix} t = \sin^2 x+1 \\ dt = 2\sin x\cos x\,dx \\ \sin x\cos x\,dx = \frac{1}{2}\,dt \end{bmatrix} = \int \frac{1}{\sqrt{t}}\cdot\frac{1}{2}\,dt = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt =$$
$$= t^{\frac{1}{2}} + C = \sqrt{\sin^2 x+1} + C,\ C \in \mathbb{R}$$

Zadatak 1.2.12. Riješite: $\displaystyle\int \frac{1+x}{1+\sqrt{x}}\,dx.$

Rješenje. 1. način

$$\int \frac{1+x}{1+\sqrt{x}}\,dx = \begin{bmatrix} t = 1+\sqrt{x} \Rightarrow x = (t-1)^2 \\ dt = \frac{1}{2\sqrt{x}}\,dx \\ dx = 2(t-1)\,dt \end{bmatrix} = \int \frac{1+(t-1)^2}{t}\cdot 2(t-1)\,dt =$$
$$= 2\int \frac{(t^2-2t+2)\,(t-1)}{t}\,dt = 2\int \frac{t^3-t^2-2t^2+2t+2t-2}{t}\,dt =$$
$$= 2\int \frac{t^3-3t^2+4t-2}{t}\,dt = 2\Big(\int t^2\,dt - 3\int t\,dt + 4\int dt - 2\int \frac{1}{t}\,dt\Big) =$$
$$= 2\left(\frac{t^3}{3} - 3\frac{t^2}{2} + 4t - 2\ln|t|\right) + C =$$
$$= \frac{2}{3}(1+\sqrt{x})^3 - 3(1+\sqrt{x})^2 + 8(1+\sqrt{x}) - 4\ln|1+\sqrt{x}| + C,\ C\in\mathbb{R}$$

2. način

$$\int \frac{1+x}{1+\sqrt{x}}\,dx = \begin{bmatrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \\ 2\sqrt{x}\,dt = dx \Rightarrow dx = 2t\,dt \end{bmatrix} = \int \frac{1+t^2}{1+t}\cdot 2t\,dt = 2\int \frac{t^3+t}{t+1}\,dt =$$
$$\overset{(*)}{=} 2\int \left(t^2 - t + 2 - \frac{2}{t+1}\right)dt =$$
$$= 2\left(\int t^2\,dt - \int t\,dt + \int 2\,dt - 2\int \frac{1}{t+1}\,dt\right) =$$
$$= 2\left(\frac{t^3}{3} - \frac{t^2}{2} + 2t - 2\ln|t+1|\right) + C =$$
$$= \frac{2}{3}\sqrt{x}^3 - x + 4\sqrt{x} - 4\ln|\sqrt{x}+1| + C,\ C\in\mathbb{R}$$

Dijeljenje polinoma (*):

$$\begin{array}{rl} & (t^3+t) \div (t+1) = t^2 - t + 2 \\ - & t^3 \quad + t^2 \\ \hline & -t^2 + t \\ - & -t^2 - t \\ \hline & 2t \\ - & 2t + 2 \\ \hline & -2 \end{array}$$

Zadatak 1.2.13. (Zadaci s kolokvija)

Integrirajte:

a) $\displaystyle\int \left(\sin^2 x + \operatorname{tg}^2 x\right)\, dx$, b) $\displaystyle\int \frac{1}{x\sqrt{2+\ln x}}\, dx$, c) $\displaystyle\int \frac{\cos(2x)}{1+3\sin(2x)}\, dx$.

Rješenje.

a)

$$\begin{aligned}\int \left(\sin^2 x + \operatorname{tg}^2 x\right)\, dx &= \int \left(\sin^2 x + \frac{\sin^2 x}{\cos^2 x}\right)\, dx = \int \sin^2 x\, dx + \int \frac{1-\cos^2 x}{\cos^2 x}\, dx = \\ &= \int \frac{1-\cos(2x)}{2}\, dx + \int \left(\frac{1}{\cos^2 x} - 1\right)\, dx = \\ &= \frac{1}{2}\left(\int dx - \int \cos(2x)\, dx\right) + \int \frac{1}{\cos^2 x}\, dx - \int dx = \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + \operatorname{tg} x - x + C = \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + \operatorname{tg} x + C,\ C \in \mathbb{R}\end{aligned}$$

b)

$$\begin{aligned}\int \frac{1}{x\sqrt{2+\ln x}}\, dx &= \begin{bmatrix} t = 2+\ln x \\ dt = \frac{1}{x}\, dx \end{bmatrix} = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\, dt = 2t^{\frac{1}{2}} + C = \\ &= 2\sqrt{2+\ln x} + C,\ C \in \mathbb{R}\end{aligned}$$

c)

$$\begin{aligned}\int \frac{\cos(2x)}{1+3\sin(2x)}\, dx &= \begin{bmatrix} t = 1+3\sin(2x) \\ dt = 3\cos(2x)\cdot 2\, dx \\ \cos(2x)\, dx = \frac{1}{6}\, dt \end{bmatrix} = \int \frac{1}{t}\cdot\frac{1}{6}\, dt = \frac{1}{6}\int \frac{1}{t}\, dt = \frac{1}{6}\ln|t| + C = \\ &= \frac{1}{6}\ln|1+3\sin(2x)| + C,\ C \in \mathbb{R}\end{aligned}$$

Trigonometrijske supstitucije

Postoje različite vrste trigonometrijskih supstitucija, no mi ćemo obraditi one najvažnije. Kod njih je ideja koristiti osnovni trigonometrijski identitet te ostale trigonometrijske formule.

1. tip:

$$\int f\left(\sqrt{k^2 - x^2}\right)\, dx = \begin{bmatrix} x = k\sin t \\ dx = k\cos t\, dt \end{bmatrix}$$

2. tip:

$$\int f\left(\sqrt{x^2-k^2}\right)\,dx = \begin{bmatrix} x = \frac{k}{\cos t} \\ dx = \frac{k\sin t}{\cos^2 t}\,dt \end{bmatrix}$$

3. tip:

$$\int f\left(\sqrt{k^2+x^2}\right)\,dx = \begin{bmatrix} x = k\,\mathrm{tg}\,t \\ dx = k\frac{1}{\cos^2 t}\,dt \end{bmatrix}$$

Zadatak 1.2.14. Koristeći trigonometrijske supstitucije, riješite integrale:

a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$, b) $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$, c) $\displaystyle\int \frac{\sqrt{x^2+1}}{x}\,dx$.

Rješenje.

a)

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} x = \sin t \Rightarrow t = \arcsin x \\ dx = \cos t\,dt \end{bmatrix} = \int \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cdot\cos t\,dt = \int \frac{\sin^2 t}{\sqrt{\cos^2 t}}\cdot\cos t\,dt =$$

$$= \int \sin^2 t\,dt = \frac{1}{2}\int (1-\cos(2t))\,dt = \begin{bmatrix} s = 2t \\ ds = 2\,dt \\ dt = \frac{1}{2}\,ds \end{bmatrix} =$$

$$= \frac{1}{2}\left(\int dt - \frac{1}{2}\int \cos s\,ds\right) = \frac{1}{2}\left(t - \frac{1}{2}\sin s\right) + C =$$

$$= \frac{1}{2}\arcsin x - \frac{1}{4}\sin(2\arcsin x) + C,\ C\in\mathbb{R}$$

b)

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \begin{bmatrix} x = \frac{2}{\cos t} \Rightarrow t = \arccos\frac{2}{x} \\ dx = \frac{2\sin t}{\cos^2 t}\,dt \end{bmatrix} = \int \frac{\frac{2}{\cos t}}{\sqrt{\frac{4}{\cos^2 t}-4}}\cdot\frac{2\sin t}{\cos^2 t}\,dt =$$

$$= \int \frac{\frac{2}{\cos t}}{\sqrt{\frac{4-4\cos^2 t}{\cos^2 t}}}\cdot\frac{2\sin t}{\cos^2 t}\,dt = 2\int \frac{2}{2\sqrt{1-\cos^2 t}}\cdot\frac{\sin t}{\cos^2 t}\,dt =$$

$$= 2\int \frac{1}{\cos^2 t}\,dt = 2\,\mathrm{tg}\,t + C = 2\,\mathrm{tg}\left(\arccos\frac{2}{x}\right) + C,\ C\in\mathbb{R}$$

c)

$$\int \frac{\sqrt{x^2+1}}{x}\,dx = \begin{bmatrix} x = \operatorname{tg} t \Rightarrow t = \operatorname{arctg} x \\ dx = \frac{1}{\cos^2 t}\,dt \end{bmatrix} = \int \frac{\sqrt{\operatorname{tg}^2 t + 1}}{\operatorname{tg} t} \cdot \frac{1}{\cos^2 t}\,dt = \int \frac{\sqrt{\frac{\sin^2 t}{\cos^2 t} + 1}}{\operatorname{tg} t \cos^2 t}\,dt =$$

$$= \int \frac{\sqrt{\frac{1}{\cos^2 t}}}{\frac{\sin t}{\cos t}\cos^2 t}\,dt = \int \frac{1}{\sin t \cos^2 t}\,dt = \int \frac{\sin^2 t + \cos^2 t}{\sin t \cos^2 t}\,dt =$$

$$= \int \left(\frac{\sin t}{\cos^2 t} + \frac{1}{\sin t}\right)\,dt = \begin{bmatrix} s = \cos t \\ ds = -\sin t\,dt \\ \sin t\,dt = -ds \end{bmatrix} = \int \frac{-ds}{s^2} + \int \frac{1}{\sin t}\,dt =$$

$$\overset{Zad.1.2.8e)}{=} -\frac{1}{s} + \ln\left|\operatorname{tg}\frac{t}{2}\right| + C = \frac{1}{\cos t} + \ln\left|\operatorname{tg}\frac{t}{2}\right| + C =$$

$$= \frac{1}{\cos(\operatorname{arctg} x)} + \ln\left|\operatorname{tg}\frac{\operatorname{arctg} x}{2}\right| + C,\ C \in \mathbb{R}$$

1.2.2 Metoda parcijalne integracije

Teorem 1.2.15. Formula parcijalne integracije

$$\int u\,dv = uv - \int v\,du$$

Zadatak 1.2.16. Metodom parcijalne integracije riješite integrale:

a) $\int x\sin x\,dx$, b) $\int x\ln x\,dx$, c) $\int x\cos x\,dx$, d) $\int x^2e^x\,dx$.

Rješenje.

a)

$$\int x \cdot \sin x\,dx = \begin{bmatrix} u = x & dv = \sin x\,dx \\ du = dx & v = \int \sin x\,dx = -\cos x \end{bmatrix} =$$

$$= x \cdot (-\cos x) - \int (-\cos x)\,dx =$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C,\ C \in \mathbb{R}$$

b)

$$\int x\ln x\,dx = \begin{bmatrix} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}\,dx & v = \int x\,dx = \frac{x^2}{2} \end{bmatrix} = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx =$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C, C \in \mathbb{R}$$

c)

$$\int x\cos x\,dx = \begin{bmatrix} u = x & dv = \cos x\,dx \\ du = dx & v = \int \cos x\,dx = \sin x \end{bmatrix} = x\sin x - \int \sin x\,dx =$$
$$= x\sin x + \cos x + C,\ C \in \mathbb{R}$$

d)

$$\int x^2 e^x\,dx = \begin{bmatrix} u = x^2 & dv = e^x\,dx \\ du = 2x\,dx & v = \int e^x\,dx = e^x \end{bmatrix} = x^2e^x - \int e^x \cdot 2x\,dx =$$
$$= x^2e^x - 2\int xe^x\,dx = \begin{bmatrix} u = x & dv = e^x \\ du = dx & v = \int e^x\,dx = e^x \end{bmatrix} =$$
$$= x^2e^x - 2\left(xe^x - \int e^x dx\right) = x^2e^x - 2xe^x + 2e^x + C,\ C \in \mathbb{R}$$

Primjer 1.2.17.

$$\int e^x \sin x\,dx = \begin{bmatrix} u = e^x & dv = \sin x\,dx \\ du = e^x\,dx & v = \int \sin x\,dx = -\cos x \end{bmatrix} =$$
$$= e^x \cdot (-\cos x) + \int \cos x \cdot e^x\,dx = \begin{bmatrix} u = e^x & dv = \cos x\,dx \\ du = e^x\,dx & v = \int \cos x\,dx = \sin x \end{bmatrix} =$$
$$= -e^x\cos x + e^x \sin x - \int \sin x \cdot e^x\,dx$$

Ako označimo $A := e^x \sin x\,dx$, imamo:

$$A = -e^x\cos x + e^x \sin x - A$$
$$2A = e^x \sin x - e^x \cos x$$
$$A = \frac{1}{2}e^x(\sin x - \cos x)\,.$$

Prema tome

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C,\ C \in \mathbb{R}.$$

Zadatak 1.2.18. Riješite integrale:

a) $\int x^2 \cos(2x)\,dx$, b) $\int x^2 \operatorname{arctg} x\,dx$, c) $\int \arcsin x\,dx$, d) $\int \ln x\,dx$.

Rješenje.

a)

$$\int x^2 \cos(2x)\,dx = \begin{bmatrix} u = x^2 & dv = \cos(2x)\,dx \\ du = 2x\,dx & v = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x) \end{bmatrix} =$$
$$= x^2 \cdot \frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2} \cdot 2x\,dx = \frac{x^2 \sin(2x)}{2} - \int x \sin(2x)\,dx =$$
$$= \begin{bmatrix} u = x & dv = \sin(2x)\,dx \\ du = dx & v = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) \end{bmatrix} =$$
$$= \frac{x^2 \sin(2x)}{2} - \left(x \cdot \left(-\frac{1}{2}\cos(2x) \right) - \int -\frac{1}{2}\cos(2x)\,dx \right) =$$
$$= \frac{x^2 \sin(2x)}{2} + \frac{1}{2} x \cos(2x) - \frac{1}{2} \int \cos(2x) =$$
$$= \frac{1}{2} x^2 \cos(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{4}\sin(2x) + C,\ C \in \mathbb{R}$$

b)

$$\int x^2 \operatorname{arctg} x\,dx = \begin{bmatrix} u = \operatorname{arctg} x & dv = x^2\,dx \\ du = \frac{1}{1+x^2}\,dx & v = \int x^2\,dx = \frac{x^3}{3} \end{bmatrix} =$$
$$= \operatorname{arctg} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2}\,dx = \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{3} \int \frac{x^2}{1+x^2}\,xdx =$$
$$= \begin{bmatrix} t = 1 + x^2 \Rightarrow x^2 = t - 1 \\ dt = 2x\,dx \\ x\,dx = \frac{1}{2}\,dt \end{bmatrix} = \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{3} \int \frac{t-1}{t} \cdot \frac{1}{2}\,dt =$$
$$= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{6} \int \frac{t-1}{t}\,dt = \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{6} \left(\int dt - \int \frac{1}{t}\,dt \right) =$$
$$= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{6} t + \frac{1}{6} \ln|t| + C =$$
$$= \frac{1}{3} x^3 \operatorname{arctg} x - \frac{1}{6}(1 + x^2) + \frac{1}{6} \ln(1 + x^2) + C,\ C \in \mathbb{R}$$

c)

$$\int \arcsin x\,dx = \begin{bmatrix} u = \arcsin x\,dx & dv = 1\,dx \\ du = \frac{1}{\sqrt{1-x^2}} & v = \int 1\,dx = x \end{bmatrix} = \arcsin x \cdot x - \int \frac{x}{\sqrt{1-x^2}}\,dx =$$
$$= \begin{bmatrix} t = 1 - x^2 \\ dt = -2x\,dx \\ x\,dx = -\frac{1}{2}\,dt \end{bmatrix} = x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{t}}\,dt = x \arcsin x + \frac{1}{2} \int t^{-\frac{1}{2}}\,dt =$$
$$= x \arcsin x + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = x \arcsin x + \sqrt{1 - x^2} + C,\ C \in \mathbb{R}$$

d)

$$\int \ln x\,dx = \begin{bmatrix} u = \ln x & dv = 1\,dx \\ du = \frac{1}{x}\,dx & v = \int 1\,dx = x \end{bmatrix} = \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx =$$
$$= x\ln x - x + C,\ C \in \mathbb{R}$$

Zadatak 1.2.19. (Zadaci s kolokvija)

Integrirajte:

a) $\int x^2 \ln x\,dx$, b) $\int \frac{x+4}{\cos^2 x}\,dx$, c) $\int (3-x)e^x\,dx$, d) $\int (2x^2-1)3^x\,dx$.

Rješenje.

a)

$$\int x^2 \ln x\,dx = \begin{bmatrix} u = \ln x & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \int x^2\,dx = \frac{x^3}{3} \end{bmatrix} = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx =$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\frac{x^3}{3} + C =$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C,\ C \in \mathbb{R}$$

b)

$$\int \frac{x+4}{\cos^2 x}\,dx = \begin{bmatrix} u = x+4 & dv = \frac{1}{\cos^2 x}\,dx \\ du = dx & v = \int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x \end{bmatrix} = (x+4)\operatorname{tg} x - \int \operatorname{tg} x\,dx \overset{Zad.1.2.8.a)}{=}$$
$$= (x+4)\operatorname{tg} x + \ln|\cos x| + C,\ C \in \mathbb{R}$$

c)

$$\int (3-x)e^x\,dx = \begin{bmatrix} u = 3-x & dv = e^x\,dx \\ du = -dx & v = \int e^x\,dx = e^x \end{bmatrix} = (3-x)e^x - \int e^x \cdot (-dx) =$$
$$= (3-x)e^x + \int e^x\,dx = (3-x)e^x + e^x + C = 4e^x - xe^x + C,\ C \in \mathbb{R}$$

d)

$$\begin{aligned}
\int \left(2x^2-1\right) 3^x\, dx &= \begin{bmatrix} u = 2x^2-1 & dv = 3^x\, dx \\ du = 4x\, dx & v = \int 3^x\, dx = \frac{3^x}{\ln 3} \end{bmatrix} = \\
&= \left(2x^2-1\right) \cdot \frac{3^x}{\ln 3} - \int \frac{3^x}{\ln 3} \cdot 4x\, dx = \\
&= \left(2x^2-1\right) \frac{3^x}{\ln 3} - \frac{4}{\ln 3} \int x3^x\, dx = \\
&= \begin{bmatrix} u = x & dv = 3^x\, dx \\ du = dx & v = \int 3^x\, dx = \frac{3^x}{\ln 3} \end{bmatrix} = \\
&= \left(2x^2-1\right) \frac{3^x}{\ln 3} - \frac{4}{\ln 3} \left(x \cdot \frac{3^x}{\ln 3} - \int \frac{3^x}{\ln 3}\, dx \right) = \\
&= \left(2x^2-1\right) \frac{3^x}{\ln 3} - \frac{4}{\ln 3} \left(\frac{x3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x\, dx \right) = \\
&= \left(2x^2-1\right) \frac{3^x}{\ln 3} - \frac{4}{\ln 3} \left(\frac{x3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x\, dx \right) = \\
&= \left(2x^2-1\right) \frac{3^x}{\ln 3} - \frac{4x3^x}{\left(\ln 3\right)^2} + \frac{4 \cdot 3^x}{\left(\ln 3\right)^3} + C,\ C \in \mathbb{R}
\end{aligned}$$

1.2.3 Integriranje racionalnih funkcija

Funkciju f nazivamo (**pravom**) **racionalnom** ako je $f(x) = \frac{P(x)}{Q(x)}$ pri čemu su P i Q polinomi takvi da je stupanj polinoma P manji od stupnja polinoma Q.

Zadatak 1.2.20. Integrirajte:

a) $\displaystyle\int \frac{x^2-5x+9}{x^2-5x+6}\, dx$, b) $\displaystyle\int \frac{x^3+x+1}{x\left(x^2+1\right)}\, dx$, c) $\displaystyle\int \frac{x+9}{\left(x-3\right)^2}\, dx$.

Rješenje.

a) Podintegralna funkcija nije prava racionalna pa najprije treba podijeliti brojnik i nazivnik.

$$\frac{x^2-5x+9}{x^2-5x+6} = 1 + \frac{3}{x^2-5x+6}$$

Nadalje, utvrđujemo tip i broj nultočki nazivnika te sukladno njima radimo rastav drugog pribrojnika na parcijalne razlomke. Budući da nazivnik ima 2 jednostruke realne nultoče ($x_1 = 3$ i $x_2 = 2$), tražimo konstante A i B takve da

$$\frac{3}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}, \text{ za svaki } x \in \mathbb{R}.$$

Stavljanjem na zajednički nazivnik dobivamo:

$$\frac{3}{x^2-5x+6} = \frac{Ax-2A+Bx-3B}{(x-3)(x-2)}$$

pa slijedi da za svaki x vrijedi

$$Ax - 2A + Bx - 3B = 3.$$

Iz prethodne relacije zapisujemo sustav:

$$\begin{aligned} A + B &= 0 \\ -2A - 3B &= 3. \end{aligned}$$

Iz sustava dobivamo da je $B = -3$ i $A = 3$. Dakle, traženi rastav je

$$\frac{3}{x^2 - 5x + 6} = \frac{3}{x - 3} - \frac{3}{x - 2}.$$

$$\begin{aligned} \int \frac{x^2 - 5x + 9}{x^2 - 5x + 6}\, dx = \int \left(1 + \frac{3}{x^2 - 5x + 6}\right) dx = \\ = \int dx + \int \frac{3}{x^2 - 5x + 6}\, dx = x + 3\int \frac{1}{x - 3}\, dx - 3\int \frac{1}{x - 2}\, dx = \\ = x + 3\ln|x - 3| - 3\ln|x - 2| + C,\ C \in \mathbb{R} \end{aligned}$$

b) Nakon provedenog množenja u nazivniku podintegralne funkcije primjećujemo da se ne radi o pravoj racionalnoj funkciji pa najprije podijelimo polinome. Jedna nultočka nazivnika je jednostruka realna $x_1 = 0$, a druge dvije su kompleksno konjugirane $x_{2,3} = \pm i$. Iz tog razloga rastav ima sljedeći oblik:

$$\begin{aligned} \frac{1}{x(x^2 + 1)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \\ &= \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2 + 1)}, \end{aligned}$$

odnosno

$$Ax^2 + A + Bx^2 + Cx = 1.$$

Dobivamo sustav:

$$\begin{aligned} A + B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}$$

iz čega odmah dobivamo da je $B = -1$.

$$\int \frac{x^3+x+1}{x(x^2+1)}\,dx = \int \frac{x^3+x+1}{x^3+x}\,dx = \int \left(1+\frac{1}{x^3+x}\right)dx = \int dx + \int \frac{1}{x^3+x}\,dx =$$

$$= x + \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx = \begin{bmatrix} t = x^2+1 \\ dt = 2x\,dx \\ x\,dx = \frac{1}{2}\,dt \end{bmatrix} = x + \ln|x| - \int \frac{1}{2t}\,dt =$$

$$= x + \ln|x| - \frac{1}{2}\ln|t| + C = x + \ln|x| - \frac{1}{2}\ln(x^2+1) + C,\ C \in \mathbb{R}$$

c) Podintegralna funkcija je prava racionalna. Nazivnik ima jednu dvostruku nultočki $x_{1,2} = 3$ zato imamo sljedeći rastav

$$\frac{x+9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \\ = \frac{Ax-3A+B}{(x-3)^2}.$$

Dobivamo sustav:

$$A = 1$$
$$-3A + B = 9$$

iz čega slijedi da je $B = 12$.

$$\int \frac{x+9}{(x-3)^2}\,dx = \int \frac{1}{x-3}\,dx + \int \frac{12}{(x-3)^2}\,dx = \begin{bmatrix} t = x-3 \\ dt = dx \end{bmatrix} =$$

$$= \int \frac{1}{t}\,dt + 12\int t^{-2}\,dt = \ln|t| - 12\cdot\frac{1}{t} + C =$$

$$= \ln|x-3| - 12\cdot\frac{1}{x-3} + C,\ C \in \mathbb{R}$$

Primjer 1.2.21. Drugo rješenje Zadatka 1.2.11.b).

$$\int \frac{1}{3^x+2}\,dx = \begin{bmatrix} t = 3^x+2 \Rightarrow 3^x = t-2 \\ dt = 3^x\cdot\ln 3\,dx \\ dx = \frac{1}{3^x\cdot\ln 3}\,dt = \frac{1}{(t-2)\ln 3}\,dt \end{bmatrix} = \int \frac{1}{t}\cdot\frac{1}{(t-2)\ln 3}\,dt = \frac{1}{\ln 3}\int \frac{1}{t(t-2)}\,dt = (*)$$

Rastav na parcijalne razlomke je:

$$\frac{1}{t(t-2)} = \frac{A}{t} + \frac{B}{t-2} = \\ = \frac{At+Bt-2A}{t(t-2)}.$$

Dobivamo sustav:

$$A + B = 0$$
$$-2A = 1$$

pa slijedi da je $A = -\frac{1}{2}$ i $B = \frac{1}{2}$. Nastavljamo s integralom.

$$(*) = \frac{1}{\ln 3}\int\left(\frac{-\frac{1}{2}}{t} + \frac{\frac{1}{2}}{t-2}\right)dt = \frac{1}{\ln 3}\left(-\frac{1}{2}\int\frac{1}{t}\,dt + \frac{1}{2}\int\frac{1}{t-2}\,dt\right) =$$
$$= \frac{1}{\ln 3}\left(-\frac{1}{2}\ln|t| + \frac{1}{2}\ln|t-2|\right) + C = -\frac{1}{2\ln 3}\ln(3^x+2) + \frac{1}{2\ln 3}\ln 3^x + C,\ C \in \mathbb{R}$$

Zadatak 1.2.22. (Zadaci s kolokvija)

Integrirajte:

$$\text{a)} \int\frac{dx}{x^2+8x}, \quad \text{b)}\int\frac{dx}{x^2+x-2}, \quad \text{c)}\int\frac{x^3+3x^2+x+1}{2x^2+x}\,dx.$$

Rješenje.

a) Rastav na parcijalne razlomke je

$$\frac{1}{x(x+8)} = \frac{A}{x} + \frac{B}{x+8} = \\ = \frac{Ax + 8A + Bx}{x(x+8)}$$

pa je $A = \frac{1}{8}$ i $B = -\frac{1}{8}$.

$$\int\frac{dx}{x^2+8x} = \int\frac{1}{x(x+8)}\,dx = \int\frac{\frac{1}{8}}{x}\,dx + \int\frac{-\frac{1}{8}}{x+8}\,dx = \frac{1}{8}\int\frac{1}{x}\,dx - \frac{1}{8}\int\frac{1}{x+8}\,dx =$$
$$= \frac{1}{8}\ln|x| - \frac{1}{8}\ln|x+8| + C,\ C \in \mathbb{R}$$

b) Nazivnik podintegralne funkcije ima jednostruke nultočke $x_1 = 1$ i $x_2 = -2$ pa je rastav:

$$\frac{1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \\ = \frac{Ax + 2A + Bx - B}{(x-1)(x+2)}.$$

$$\int\frac{dx}{x^2+x+2} = \int\frac{\frac{1}{3}}{x-1}\,dx + \int\frac{-\frac{1}{3}}{x+2}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C,\ C \in \mathbb{R}$$

c) Podintegralna funkcija nije prava racionalna pa je najprije potrebno podijeliti polinome.

$$\begin{array}{rl} & (x^3 + 3x^2 + x + 1) \div (2x^2 + x) = \frac{1}{2}x + \frac{5}{4} \\ - & x^3 + \frac{1}{2}x^2 \\ \hline & \frac{5}{2}x^2 + x + 1 \\ - & \frac{5}{2}x^2 + \frac{5}{4}x \\ \hline & -\frac{1}{4}x + 1 \end{array}$$

Prema tome, slijedi da je:

$$\frac{x^3+3x^2+x+1}{2x^2+x} = \frac{1}{2}x + \frac{5}{4} + \frac{-\frac{1}{4}x+1}{2x^2+x}.$$

Sada je još potrebno napraviti rastav na parcijalne razlomke koji je

$$\begin{aligned}\frac{-\frac{1}{4}x+1}{2x^2+x} &= \frac{A}{x} + \frac{B}{2x+1} = \\ &= \frac{2Ax + A + Bx}{2x^2+x}\end{aligned}$$

te rješavanjem sustava:

$$\begin{aligned}2A + B &= -\frac{1}{4} \\ A &= 1\end{aligned}$$

dobijemo da je $A = 1$ i $B = -\frac{9}{4}$.

$$\begin{aligned}\int \frac{x^3+3x^2+x+1}{x(2x+1)}\,dx &= \int \left(\frac{1}{2}x + \frac{5}{4} + \frac{-\frac{1}{4}x+1}{2x^2+x}\right)dx = \\ &= \frac{1}{2}\int x\,dx + \frac{5}{4}\int dx + \int \frac{-\frac{1}{4}x+1}{x(2x+1)}\,dx = \\ &= \frac{1}{2}\cdot\frac{x^2}{2} + \frac{5}{4}x + \int \left(\frac{1}{x} - \frac{9}{4}\cdot\frac{1}{2x+1}\right)dx = \\ &= \frac{1}{4}x^2 + \frac{5}{4}x + \int \frac{1}{x}\,dx - \frac{9}{4}\int \frac{1}{2x+1}\,dx = \\ &= \frac{1}{4}x^2 + \frac{5}{4}x + \ln|x| - \frac{9}{8}\ln|2x+1| + C,\ C \in \mathbb{R}\end{aligned}$$