Poglavlje 4

Metode računanja određenog integrala

Kao i kod neodređenih integrala, kod računanja određenih integrala neelementarnih funkcija koristimo metodu supstitucije i parcijalne integracije. No, moramo voditi računa o granicama integracije.

4.1 Supstitucija u određenom integralu

Zadatak 4.1.1. Izračunajte $\displaystyle\int_0^2 2x\left(x^2+1\right)^3\,dx$.

Rješenje. 1. način: odredi se neodređeni integral pa se iskoristi Newton-Leibnizova formula.

$$\int 2x\left(x^2+1\right)^3\,dx = \begin{bmatrix} t = x^2+1 \\ dt = 2x\,dx \end{bmatrix} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{(x^2+1)^4}{4} + C,\ C \in \mathbb{R}$$

$$\int_0^2 2x\left(x^2+1\right)^3\,dx = \frac{(x^2+1)^4}{4} + C\,\Bigg|_0^2 = \frac{5^4}{4} + C - \frac{1^4}{4} - C = 156$$

2. način: sukladno uvedenoj supstituciji promijene se i granice integracije.

$$\int_0^2 2x\left(x^2+1\right)^3\,dx = \begin{bmatrix} t = x^2+1 & x_1 = 0 \Rightarrow t_1 = 0^2+1 = 1 \\ dt = 2x\,dx & x_2 = 2 \Rightarrow t_2 = 2^2+1 = 5 \end{bmatrix} =$$
$$= \int_1^5 t^3\,dt = \frac{t^4}{4}\Bigg|_1^5 = \frac{625}{4} - \frac{1}{4} = 156$$

Napomena 4.1.2. Drugi način rješavanja prošlog zadatka je bolji i ubuduće ćemo ga koristiti. Primijetite da u predzadnjem koraku nije potrebno vratiti ono što je bila supstitucija ($t = x^2 + 1$) jer smo izvršivši supstituciju na opisani načina, u potpunosti "preveli" integriranje po x u integiranje po t. Također, veza početne varijable x i supstitucijske varijable t mora biti jednoznačna.

Zadatak 4.1.3. Izračunajte: $\displaystyle\int_1^4 \frac{x}{\sqrt{1+2x}}\,dx.$

Rješenje.

$$\int_1^4 \frac{x}{\sqrt{1+2x}}\,dx = \begin{bmatrix} t = 1+2x \Rightarrow x = \frac{t-1}{2} \\ dt = 2\,dx \quad x_1 = 1 \Rightarrow t_1 = 3 \\ dx = \frac{1}{2}\,dt \quad x_2 = 4 \Rightarrow t_2 = 9 \end{bmatrix} = \int_3^9 \frac{\frac{t-1}{2}}{\sqrt{t}} \cdot \frac{1}{2}\,dt = \frac{1}{4}\int_3^9 \frac{t-1}{\sqrt{t}}\,dt =$$

$$= \frac{1}{4}\int_3^9 \left(\sqrt{t} - t^{-\frac{1}{2}}\right)\,dt = \frac{1}{4}\left(\frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)\Bigg|_3^9 =$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}\sqrt{9}^3 - 2\sqrt{9}\right) - \left(\frac{2}{3}\sqrt{3}^3 - 2\sqrt{3}\right)\right] = 3$$

Zadatak 4.1.4. Izračunajte: $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\,dx.$

Rješenje.

$$\int_0^{\ln 2} \sqrt{e^x-1}\,dx = \begin{bmatrix} t^2 = e^x - 1 & e^x = t^2+1 \\ 2t\,dt = e^x\,dx & x_1 = 0 \Rightarrow t^2 = e^0 - 1 = 0 \Rightarrow t_1 = 0 \\ 2t\,dt = (t^2+1)\,dx & x_2 = \ln 2 \Rightarrow t^2 = e^{\ln 2} - 1 = 1 \Rightarrow \\ dx = \frac{2t}{t^2+1}\,dt & t_2 = \pm 1 \Rightarrow t_2 = 1 \end{bmatrix} =$$

$$= \int_0^1 \sqrt{t^2} \cdot \frac{2t}{t^2+1}\,dt = 2\int_0^1 |t| \cdot \frac{t}{t^2+1}\,dt = 2\int_0^1 \frac{t^2}{t^2+1}\,dt =$$

$$= 2\int_0^1 \frac{t^2+1-1}{t^2+1}\,dt = 2\left(\int_0^1 1\,dt - \int_0^1 \frac{1}{t^2+1}\,dt\right) =$$

$$= 2\,(t - \arctan t)\Bigg|_0^1 = 2\left(1 - \frac{\pi}{4}\right)$$

Napomena 4.1.5. Da smo u supstituciji izabrali za drugu granicu $t_2 = -1$, onda bi interval integracije bio $[-1, 0]$ te bismo imali:

$$\dots = \int_0^{-1} |t| \cdot \frac{2t}{t^2+1}\,dt = \int_0^{-1} -t \cdot \frac{2t}{t^2+1}\,dt = -\int_{-1}^0 -t \cdot \frac{2t}{t^2+1}\,dt = 2\int_{-1}^0 \frac{t^2}{t^2+1}\,dt = \dots$$

Napomena 4.1.6. Ovdje se radi o integralu parne funkcije za koju se može pokazati da za $a > 0$ vrijedi $\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$ zbog simetrije grafa takve funkcije s obzirom na os y. Štoviše, iz tog svojstva također slijedi: $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Zadatak 4.1.7. Izračunajte: $\int_{-\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

Rješenje.

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \begin{bmatrix} x = \sin t & x_1 = -\frac{\sqrt{2}}{2} \Rightarrow \sin t = -\frac{\sqrt{2}}{2} \Rightarrow t_1 = -\frac{\pi}{4} \\ dx = \cos t\,dt & x_2 = \frac{1}{2} \Rightarrow \sin t = \frac{1}{2} \Rightarrow t_2 = \frac{\pi}{6} \end{bmatrix} =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\cdot\cos t\,dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} |\cos t|\cdot\cos t\,dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \cos^2 t\,dt =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} (1+\cos(2t))\,dt =$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{5\pi}{12} + \frac{\sqrt{3}}{4} + \frac{1}{2}\right)$$

U četvrtoj jednakosti maknuli smo apsolutnu vrijednost jer funkcija $g(t) = \cos t$ ima pozitivne vrijednosti za $t \in \left\langle -\frac{\pi}{4}, \frac{\pi}{6}\right\rangle$ (na intervalu integracije).

Zadatak 4.1.8. Izračunajte: $\int_{1}^{e} \frac{\sin(\ln x)}{x}\,dx$.

Rješenje.

$$\int_{1}^{e} \frac{\sin(\ln x)}{x}\,dx = \begin{bmatrix} t = \ln x & x_1 = 1 \Rightarrow t_1 = \ln 1 = 0 \\ dt = \frac{1}{x}\,dx & x_2 = e \Rightarrow t_2 = \ln e = 1 \end{bmatrix} = \int_{0}^{1} \sin t\,dt = -\cos t\Bigg|_{0}^{1} =$$

$$= -\cos 1 - (-\cos 0) = -\cos 1 + 1$$

4.2 Parcijalna integracija u određenom integralu

Formula: $\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du.$

Zadatak 4.2.1. Riješite: $\int_{\ln 1}^{\ln 2} xe^x\,dx$.

Rješenje.

$$\int_{\ln 1}^{\ln 2} xe^x\,dx = \begin{bmatrix} u = x & dv = e^x\,dx \\ du = dx & v = \int e^x\,dx = e^x \end{bmatrix} = xe^x\Big|_{\ln 1}^{\ln 2} - \int_{\ln 1}^{\ln 2} e^x\,dx =$$

$$= \left(\ln 2 \cdot e^{\ln 2} - \ln 1 \cdot e^{\ln 1}\right) - \int_{\ln 1}^{\ln 2} e^x\,dx = 2\ln 2 - 0 - e^x\Big|_{\ln 1}^{\ln 2} =$$

$$= 2\ln 2 - \left(e^{\ln 2} - e^{\ln 1}\right) = 2\ln 2 - 1$$

Zadatak 4.2.2. Riješite: $\int_0^{\frac{\pi}{2}} (x+3)\sin x\,dx.$

Rješenje.

$$\int_0^{\frac{\pi}{2}} (x+3)\sin x\,dx = \begin{bmatrix} u = x+3 & dv = \sin x\,dx \\ du = dx & v = \int \sin x\,dx = -\cos x \end{bmatrix} =$$

$$= (x+3)\cdot(-\cos x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\,dx =$$

$$= \left(\frac{\pi}{2} + 3\right)\cdot\left(-\cos\frac{\pi}{2}\right) - 3\cdot(-\cos 0) + \int_0^{\frac{\pi}{2}} \cos x\,dx =$$

$$= 3 + \sin x\Big|_0^{\frac{\pi}{2}} = 3 + \sin\frac{\pi}{2} - \sin 0 = 4$$

Zadatak 4.2.3. Riješite: $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,dx.$

Rješenje.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,dx = \begin{bmatrix} u = x & dv = \frac{1}{\sin^2 x}\,dx \\ du = dx & v = \int \frac{1}{\sin^2 x}\,dx = -\operatorname{ctg} x \end{bmatrix} = x\cdot(-\operatorname{ctg} x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\operatorname{ctg} x\,dx =$$

$$= \frac{\pi}{3}\cdot\left(-\operatorname{ctg}\frac{\pi}{3}\right) - \frac{\pi}{4}\cdot\left(-\operatorname{ctg}\frac{\pi}{4}\right) + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx =$$

$$= \begin{bmatrix} t = \sin x & x_1 = \frac{\pi}{4} \Rightarrow t_1 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ dt = \cos x\,dx & x_2 = \frac{\pi}{3} \Rightarrow t_2 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{bmatrix} = -\frac{\sqrt{3}\pi}{9} + \frac{\pi}{4} + \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t}\,dt =$$

$$= -\frac{\sqrt{3}\pi}{9} + \frac{\pi}{4} + \ln|t|\Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}\pi}{9} + \frac{\pi}{4} + \ln\frac{\sqrt{3}}{\sqrt{2}}$$

Zadatak 4.2.4. Riješite: $\int_0^1 \operatorname{arctg} x\,dx.$

Rješenje.

$$\int_0^1 \operatorname{arctg} x\,dx = \begin{bmatrix} u = \operatorname{arctg} x & dv = 1\,dx \\ du = \frac{1}{1+x^2}\,dx & v = \int 1\,dx = x \end{bmatrix} = \operatorname{arctg} x \cdot x \Big|_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\,dx =$$

$$= \operatorname{arctg} 1 \cdot 1 - \operatorname{arctg} 0 \cdot 0 - \int_0^1 \frac{x}{1+x^2}\,dx =$$

$$= \begin{bmatrix} t = 1+x^2 & x_1 = 0 \Rightarrow t_1 = 1+0^2 = 1 \\ dt = 2x\,dx & x_2 = 1 \Rightarrow t_2 = 1+1^2 = 2 \\ x\,dx = \frac{1}{2}\,dt & \end{bmatrix} = \frac{\pi}{4} - \int_1^2 \frac{1}{t} \cdot \frac{1}{2}\,dt =$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln|t| \Big|_1^2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Zadatak 4.2.5. Riješite: $\displaystyle\int_0^{e-1} \ln(1+x)\,dx.$

Rješenje.

$$\int_0^{e-1} \ln(1+x)\,dx = \begin{bmatrix} t = 1+x & x_1 = 0 \Rightarrow t_1 = 1+0 = 1 \\ dt = dx & x_2 = e-1 \Rightarrow t_2 = 1+e-1 = e \end{bmatrix} = \int_1^e \ln t\,dt =$$

$$= \begin{bmatrix} u = \ln t & dv = 1\,dt \\ du = \frac{1}{t}\,dt & v = \int 1\,dt = t \end{bmatrix} = \ln t \cdot t \Big|_1^e - \int_1^e t \cdot \frac{1}{t}\,dt =$$

$$= e \cdot \ln e - 1 \cdot \ln 1 - t \Big|_1^e = e - (e-1) = 1$$

4.3 Nepravi integral

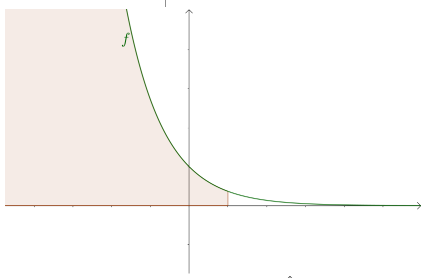
Nepravi integral je vrsta određenog integrala kod kojeg je jedna ili obje granice integracije $\pm\infty$ ili unutar područja integracije postoji točka/točke koje nisu u domeni funkcije $f(x)$ koja se integrira. Razlikujemo 4 tipa nepravog integrala. S desne strane, pored svakog od tipova dan je jedan primjer funkcije koja odgovara nepravom integralu. Kod prva tri grafa radi se o pozitivnim funkcijama pa pripradni integrali imaju istu geometrijsku intepretaciju kao ranije - površina ispod grafa funkcije. Pitanje koje se nameće je kolike su te površine budući da su u nekom smislu "površine do beskonačnosti". Naime, rezultat nepravog integrala može biti konačan broj ili beskonačno pa kažemo da integral konvergira ili divergira.

1. tip:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

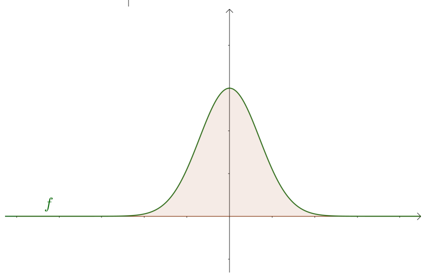
2. tip:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$



3. tip:

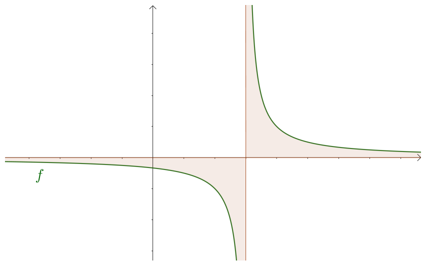
$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to+\infty}\int_c^b f(x)\,dx$$



4. tip:

Ako $f : [a,b] \to \mathbb{R}$ nije definirana u $c \in [a,b]$:

$$\int_a^b f(x)\,dx = \lim_{c_1\to c^-}\int_a^{c_1} f(x)\,dx + \lim_{c_2\to c^+}\int_{c_2}^b f(x)\,dx,$$



pri čemu je $\lim_{c_1\to c^-}$ limes k c slijeva, a $\lim_{c_2\to c^+}$ limes k c zdesna.

Zadatak 4.3.1. Riješite: $\displaystyle\int_1^{+\infty}\frac{1}{x}\,dx.$

Rješenje.

$$\int_1^{+\infty}\frac{1}{x}\,dx = \lim_{b\to+\infty}\int_1^b\frac{1}{x}\,dx = \lim_{b\to+\infty}\ln|x|\Big|_1^b = \lim_{b\to+\infty}(\ln|b| - \ln 1) = +\infty$$

Zadatak 4.3.2. Riješite: $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx.$

Rješenje.

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{+\infty} \frac{1}{1+x^2}\,dx =$$
$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to+\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx =$$
$$= \lim_{a\to-\infty} \operatorname{arctg} x \Big|_{a}^{0} + \lim_{b\to+\infty} \operatorname{arctg} x \Big|_{0}^{b} =$$
$$= \lim_{a\to-\infty} (\operatorname{arctg} 0 - \operatorname{arctg} a) + \lim_{b\to+\infty} (\operatorname{arctg} b - \operatorname{arctg} 0) =$$
$$= \left(0 - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

Napomena 4.3.3. U prvom koraku, kod raspisa početnog integrala na dva integrala, mogli smo koristiti neku drugu novu granicu koju smo postavili kao gornju granicu prvog integrala, odnosnu donju drugog integrala. Zbog prirode podintegralne funkcije (parna funkcija), nula se činila kao najbolji izbor.

Napomena 4.3.4. Funkcija $g(x) = \operatorname{tg} x$ nije definirana za $x = \pm\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$. Kada računamo tangens za x-eve koju su blizu navedenim vrijednostima, dobivamo sve veće vrijednosti pa je sukladno tome inverzna funkcija $\operatorname{arctg} x$ za "velike" x-eve jednaka $\frac{\pi}{2}$. Analogno zaključivanje je za x-eve koji teže k $-\infty$.

Zadatak 4.3.5. Riješite: $\displaystyle\int_{1}^{4} \frac{1}{\sqrt[3]{(x-2)^2}}\,dx.$

Rješenje. Podintegralna funkcija nije definirana u $x = 2$ pa se radi o 4. tipu nepravog integrala.

$$\int_{1}^{4} \frac{1}{\sqrt[3]{(x-2)^2}}\,dx = \int_{1}^{2} \frac{1}{\sqrt[3]{(x-2)^2}}\,dx + \int_{2}^{4} \frac{1}{\sqrt[3]{(x-2)^2}}\,dx =$$
$$= \lim_{c_1\to2^-} \int_{1}^{c_1} (x-2)^{-\frac{2}{3}}\,dx + \lim_{c_2\to2^+} \int_{c_2}^{4} (x-2)^{-\frac{2}{3}}\,dx =$$
$$= \lim_{c_1\to2^-} \frac{(x-2)^{\frac{1}{3}}}{\frac{1}{3}} \Bigg|_{1}^{c_1} + \lim_{c_2\to2^+} \frac{(x-2)^{\frac{1}{3}}}{\frac{1}{3}} \Bigg|_{c_2}^{4} =$$
$$= \lim_{c_1\to2^-} 3\left((c_1-2)^{\frac{1}{3}} - (1-2)^{\frac{1}{3}}\right) + \lim_{c_2\to2^+} 3\left((4-2)^{\frac{1}{3}} - (c_2-2)^{\frac{1}{3}}\right) =$$
$$= \lim_{c_1\to2^-} 3\left(\sqrt[3]{c_1-2} - \sqrt[3]{-1}\right) + \lim_{c_2\to2^+} 3\left(\sqrt[3]{2} - \sqrt[3]{c_2-2}\right) =$$
$$= 3(\sqrt[3]{2-2}+1) + 3(\sqrt[3]{2} - \sqrt[3]{2-2}) = 3\sqrt[3]{2} + 3$$

Napomena 4.3.6. U prošlom zadatku mogli smo izvršiti i supstituciju kod određenog integrala. Raspis nakon 3. jednakosti bio bi:

$$(\dots) = \begin{bmatrix} t = x-2 & x_1 = 1 \Rightarrow t_1 = -1 & x_1 = c_2 \Rightarrow t_1 = c_2 - 2 \\ dt = dx & x_2 = c_1 \Rightarrow t_2 = c_1 - 2 & x_2 = 4 \Rightarrow t_2 = 4 - 2 = 2 \end{bmatrix} =$$

$$= \lim_{c_1 \to 2^-} \int_1^{c_1-2} t^{-\frac{2}{3}}\, dt + \lim_{c_2 \to 2^+} \int_{c_2-2}^{2} t^{-\frac{2}{3}}\, dt = \lim_{c_1 \to 2^-} \left.\frac{t^{\frac{1}{3}}}{\frac{1}{3}}\right|_{-1}^{c_1-2} + \lim_{c_2 \to 2^+} \left.\frac{t^{\frac{1}{3}}}{\frac{1}{3}}\right|_{c_2-2}^{2} = (\dots)$$

Zadatak 4.3.7. Riješite: $\displaystyle\int_0^{+\infty} xe^{-x}\, dx.$

Rješenje.

$$\int_0^{+\infty} xe^{-x}\, dx = \lim_{b \to +\infty} \int_0^b xe^{-x}\, dx = \begin{bmatrix} u = x & dv = e^{-x}\, dx \\ du = dx & v = \int e^{-x}\, dx = -e^{-x} \end{bmatrix} =$$

$$= \lim_{b \to +\infty} \left(-xe^{-x} \Big|_0^b - \int_0^b (-e^{-x})\, dx \right) = \lim_{b \to +\infty} \left(-be^{-b} + \int_0^b e^{-x}\, dx \right) =$$

$$= \lim_{b \to +\infty} \left(-be^{-b} + (-e^{-x}) \Big|_0^b \right) = \lim_{b \to +\infty} \left(-be^{-b} - e^{-b} + 1 \right) =$$

$$= \lim_{b \to +\infty} \left(-\frac{b}{e^b} - e^{-b} + 1 \right) = 1$$

U zadnjem koraku koristili smo L'Hospitalovo pravilo:

$$\lim_{b \to +\infty} \frac{-b}{e^b} = \left(\frac{-\infty}{+\infty} \right) \overset{LH}{=} \lim_{b \to +\infty} \frac{-1}{e^b} = 0.$$

Zadatak 4.3.8. Riješite: $\displaystyle\int_e^{+\infty} \frac{1}{x \ln^2 x}\, dx.$

Rješenje.

$$\int_e^{+\infty} \frac{1}{x \ln^2 x}\, dx = \lim_{b \to +\infty} \int_e^b \frac{1}{x \ln^2 x}\, dx = \begin{bmatrix} t = \ln x & x_1 = e \Rightarrow t_1 = \ln e = 1 \\ dt = \frac{1}{x}\, dx & x_2 = b \Rightarrow t_2 = \ln b \end{bmatrix} =$$

$$= \lim_{b \to +\infty} \int_1^{\ln b} \frac{1}{t^2}\, dt = \lim_{b \to +\infty} \left.\frac{t^{-1}}{-1}\right|_1^{\ln b} = \lim_{b \to +\infty} \left(-\frac{1}{\ln b} + 1 \right) = 1$$

Zadatak 4.3.9. Riješite: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}.$

Rješenje. Budući da $x = 0$ nije u domeni podintegralne funkcije, zadatku moramo pristupiti na sljedeći način.

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0} \int_a^1 x^{-\frac{1}{2}}\, dx = \lim_{a \to 0} \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_a^1 = \lim_{a \to 0} \left(\frac{1^{\frac{1}{2}}}{\frac{1}{2}} - \frac{a^{\frac{1}{2}}}{\frac{1}{2}} \right) = 2$$

Zadatak 4.3.10. (Zadaci s kolokvija)

Riješite:

a) $\displaystyle\int_{-\infty}^{-3} 3^{2x-1}\,dx,$ b) $\displaystyle\int_{-\infty}^{0} e^{2x-1}\,dx.$

Rješenje.

a)

$$\int_{-\infty}^{-3} 3^{2x-1}\,dx = \lim_{a\to-\infty}\int_{a}^{-3} 3^{2x-1}\,dx = \lim_{a\to-\infty}\int_{a}^{-3} 9^{x}\cdot 3^{-1}\,dx = \frac{1}{3}\lim_{a\to-\infty}\int_{a}^{-3} 9^{x}\,dx =$$

$$= \frac{1}{3}\lim_{a\to-\infty}\frac{9^{x}}{\ln 9}\Bigg|_{a}^{-3} = \frac{1}{3}\lim_{a\to-\infty}\left(\frac{9^{-3}}{\ln 9}-\frac{9^{a}}{\ln 9}\right) = \frac{1}{3}\left(\frac{9^{-3}}{\ln 9}-0\right) = \frac{1}{3^{7}\ln 9}$$

b)

$$\int_{-\infty}^{0} e^{2x-1}\,dx = \lim_{a\to-\infty}\int_{a}^{0} e^{2x}\cdot e^{-1}\,dx = \frac{1}{e}\lim_{a\to-\infty}\int_{a}^{0} e^{2x}\,dx =$$

$$= \begin{bmatrix} t=2x & x_1=a\Rightarrow t_1=2a \\ dt=2\,dx & x_2=0\Rightarrow t_2=0 \\ dx=\frac{1}{2}\,dt & \end{bmatrix} = \frac{1}{e}\lim_{a\to-\infty}\int_{2a}^{0} e^{t}\cdot\frac{1}{2}\,dt =$$

$$= \frac{1}{2e}\lim_{a\to-\infty}\int_{2a}^{0} e^{t}\,dt = \frac{1}{2e}\lim_{a\to-\infty} e^{t}\Bigg|_{2a}^{0} = \frac{1}{2e}\lim_{a\to-\infty}(e^{0}-e^{2a}) =$$

$$= \frac{1}{2e}(1-0) = \frac{1}{2e}$$