MATEMATIKA 1

KOLOKVIJI 2003./04. - RJEŠENJA

3. kolokvij

3. KOLOKVIJ – MATEMATIKA 1 – RJEŠENJA , grupa A

1. Nađite sljedeće limese bez L'Hospitalovog pravila:

a) $\lim\limits_{x\to\infty}(\sqrt{x^2-5x+6}-x)$

b) $\lim\limits_{x\to 0}\dfrac{x-\sin 2x}{x+\sin 3x}$

c) $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x-3}\right)^{x+2}$

Rješenje. a)

$$\lim_{x\to\infty}(\sqrt{x^2-5x+6}-x)=\lim_{x\to\infty}(\sqrt{x^2-5x+6}-x)\cdot\frac{\sqrt{x^2-5x+6}+x}{\sqrt{x^2-5x+6}+x}=$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-5x+6-\cancel{x^2}}{\sqrt{x^2-5x+6}+x}=\lim_{x\to\infty}\frac{6-5x}{\sqrt{x^2-5x+6}+x}\ /:x=\lim_{x\to\infty}\frac{\frac{6}{x}-5}{\sqrt{1-\frac{5}{x}+\frac{6}{x^2}}+1}$$

$$=\boxed{\frac{-5}{2}}$$

b)

$$\lim_{x\to 0}\frac{x-\sin 2x}{x+\sin 3x}=\lim_{x\to 0}\frac{1-\frac{\sin 2x}{2x}\cdot 2}{1+\frac{\sin 3x}{3x}\cdot 3}=\frac{1-2}{1+3}=\boxed{\frac{-1}{4}}$$

c)

$$\lim_{x\to\infty}\left(\frac{x-1}{x+3}\right)^{x+2}=e^{\lim\limits_{x\to\infty}\left(\frac{x-1}{x+3}-1\right)\cdot(x+2)}=e^{\lim\limits_{x\to\infty}\left(\frac{-4}{x+3}\cdot(x+2)\right)}=$$

$$=e^{-4\lim\limits_{x\to\infty}\frac{x+2}{x+3}}\ /:x=e^{-4\lim\limits_{x\to\infty}\frac{1+\frac{2}{x}}{1+\frac{3}{x}}}=e^{-4\cdot 1}=\underline{e^{-4}}$$

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2. Derivirajte zadane funkcije:

a) $f(x)=\arctan\frac{1+x}{1-x}+\sin 2x\cdot\ln(1-x^2)$

b) $\operatorname{tg} 2y=xy$

c) $f(x)=x^{\sin 2x}$

Rješenje. a)

$$f'(x)=\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\cdot\left(\frac{1+x}{1-x}\right)'+2\cos 2x\cdot\ln(1-x^2)+\sin 2x\cdot\frac{1}{1-x^2}\cdot(-2x)=$$

$$=\frac{\cancel{(1-x)^2}}{(1-x)^2+(1+x)^2}\cdot\frac{1\cdot(1-x)-(1+x)\cdot(-1)}{\cancel{(1-x)^2}}+2\cos 2x\cdot\ln(1-x^2)-\frac{2x}{1-x^2}\sin 2x=$$

$$=\frac{1-\cancel{x}+1+\cancel{x}}{1+x^2-\cancel{2x}+1+x^2+\cancel{2x}}+2\cos 2x\ln(1-x^2)-\frac{2x}{1-x^2}\sin 2x=$$

$$=\boxed{\frac{1}{x^2+1}+2\cos 2x\cdot\ln(1-x^2)-\frac{2x}{1-x^2}\sin 2x}$$

b) $\operatorname{tg}2y=xy\ |'$

$$\frac{1}{\cos^2 2y}\cdot 2y'=1\cdot y+x\cdot y'\Rightarrow y'\left(\frac{2}{\cos^2 2y}-x\right)=y\Rightarrow y'=\boxed{\frac{y\cdot\cos^2 2y}{2-x\cos^2 2y}}$$

c) $f(x)=x^{\sin 2x}\ |\ln\Rightarrow\ \ln f(x)=\ln x^{\sin 2x}$

$\ln f(x)=\sin 2x\cdot\ln x\ |'\ \Rightarrow$

$$\frac{1}{f(x)} f'(x) = 2\cos 2x \ln x + \sin 2x \cdot \frac{1}{x}$$

$$f'(x) = x^{\sin 2x} \left(2\cos 2x \ln x + \sin 2x \cdot \frac{1}{x} \right)$$

3. Zadana je funkcija $f(x) = \sqrt{3} \arcsin(\sqrt{x+1})$.

(a) Nađite domenu od f.

(b) Nađite tangentu na graf te funkcije u točki gdje graf siječe pravac $x = -\frac{1}{2}$.

(c) Odredite kut između normale na tu tangentu i x-osi.

Rješenje. (a) $-1 \le \sqrt{x+1} \le 1$ (zbog arcsin funkcije)

$x+1 \geq 0$ (zbog $\sqrt{}$ funkcije)

$\Rightarrow \underline{x \geq -1}$

$-1 \le \sqrt{x+1}$ vrijedi za sve $x \in \mathbb{R}$

$\sqrt{x+1} \le 1 \;/\; ()^2 \Rightarrow \quad x+1 \le 1 \Rightarrow \underline{x \le 0}$

$\Rightarrow \boxed{\mathcal{D}(f) = [-1, 0]}$

(b) $x_0 = -\frac{1}{2} \Rightarrow \quad f(-\frac{1}{2}) = \sqrt{3} \arcsin \sqrt{-\frac{1}{2}+1} = \sqrt{3} \arcsin \sqrt{\frac{1}{2}} = \sqrt{3} \arcsin \frac{\sqrt{2}}{2} = \sqrt{3} \cdot \frac{\pi}{4}$

$\Rightarrow$ povlačimo tangentu u $\left(-\frac{1}{2}, \frac{\sqrt{3}}{4}\pi\right)$

$$f'(x) = \sqrt{3} \cdot \frac{1}{\sqrt{1-(\sqrt{x+1})^2}} \cdot (\sqrt{x+1})' = \frac{\sqrt{3}}{\sqrt{\cancel{1}-x-\cancel{1}}} \cdot \frac{1}{2\sqrt{x+1}} = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{-x(x+1)}}$$

$$f'(-\tfrac{1}{2}) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{2}}} = \frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{2} = \sqrt{3}$$

$y - f(x_0) = f'(x_0)(x - x_0)$, pa je tangenta $y = \sqrt{3}(x + \frac{1}{2}) + \frac{\sqrt{3}}{4}\pi$

$$y = \sqrt{3}x + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\pi$$

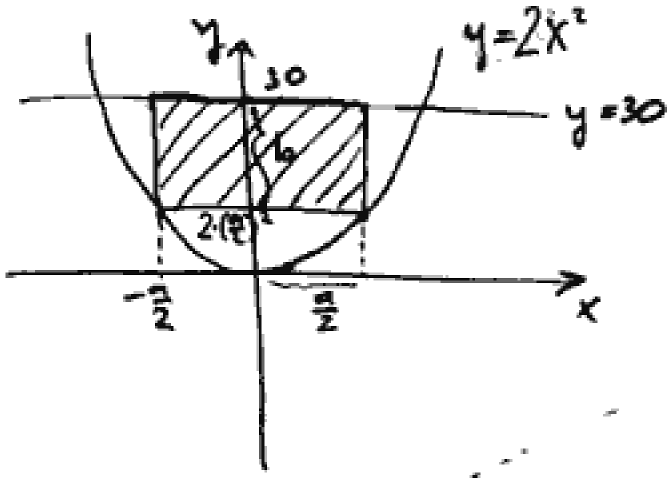
$$\underline{y = \sqrt{3}x + \frac{\sqrt{3}}{4}(2+\pi)}$$

(c) koeficijent smjera normale je $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$, pa za kut α između normale i x-osi imamo $\operatorname{tg}\alpha = -\frac{\sqrt{3}}{3} \Rightarrow \underline{\alpha = \frac{5\pi}{6}}$ Kako je $\alpha > \frac{\pi}{2}$, to je kut $\underline{\frac{\pi}{6}}$

4. (a) Odsječku parabole $y = 2x^2$ odsječenom pravcem $y = 30$ upišite pravokutnik maksimalne površine čija jedna stranica leži na tom pravcu.

(b) Koristeći linearnu aproksimaciju i dobivenu vrijednost za stranicu pravokutnika u (a) izračunajte približno površinu maksimalnog pravokutnika i usporedite s točnom vrijednošću izračunatom u (a) zadatku.

Rješenje. Slika:



$P=a\cdot b$, gdje su a i b duljine stranica pravokutnika, a za b vrijedi:

$$b = 30 - 2\cdot\left(\frac{a}{2}\right)^2 = 30 - \frac{1}{2}a^2$$

$$\Rightarrow P(a) = a\cdot\left(30-\frac{1}{2}a^2\right)$$

$$P'(a) = 30 - \frac{1}{2}a^2 + a\cdot(-a) = 30 - \frac{3}{2}a^2 = 0$$

$$60 = 3a^2$$

$$a^2 = 20 \Rightarrow \underline{a = 2\sqrt{5}}$$

$P''(a) = -3a \Rightarrow P''(5\sqrt{2})<0 \Rightarrow$ radi se o točki u kojoj se postiže lok. maksimum

$\Rightarrow b = 30-\frac{1}{2}\cdot 20 = 20$, pa je riječ o pravokutniku sa stranicama duljina $2\sqrt{5}$ i 20 (koordinate vrhova su $(\sqrt{5},10), (-\sqrt{5},10), (\sqrt{5},30), (-\sqrt{5},30)$), a maks. površina glasi $40\sqrt{5}$.

b) $P(a) = 30a - \frac{1}{2}a^3$, $a = 2\sqrt{5} = a_0 + \Delta a \approx 4.47$

$\Rightarrow a_0 = 4.5 = \frac{9}{2}$, $\Delta a = 0.03 = \frac{3}{100}$

$$P(2\sqrt{5}) \approx P(a_0) + \Delta a\cdot P'(a_0) = 30\cdot\frac{9}{2} - \frac{1}{2}\cdot\left(\frac{9}{2}\right)^3 - \frac{3}{100}\cdot\left(30-\frac{3}{2}\cdot\left(\frac{9}{2}\right)^2\right) =$$

$$= 135 - \frac{1}{2}\cdot\frac{729}{8} - \frac{3}{100}\left(30-\frac{3}{2}\cdot\frac{81}{4}\right) = 135 - \frac{729}{16} - \frac{3}{100}\cdot\frac{-(3)}{8}$$

$$= \frac{108000-36450+9}{800} = \frac{71559}{800} \approx 89.44875$$, dok je

stvarna površina $P(2\sqrt{5}) = 40\sqrt{5} \approx 89.44271$.

Napomena: moglo se uzeti i $a_0=4$ ili $a_0=5$, ali onda se dobiva slabija aproksimacija. Također, površina je mogla biti izražena kao funkcija polovine stranice a ili funkcija stranice b, pa bi rješenje izgledalo drukčije.

5. Nacrtajte graf funkcije $f(x) = \frac{x^2-5x+7}{x-2}$ (ne morate tražiti točke infleksije)

Rješenje. 1) $\mathcal{D}(f) = \mathbb{R}\setminus\{2\}$

2) $N(f) = ?$ $\quad x^2-5x+7=0$

$x_{1,2} = \frac{5\pm\sqrt{25-28}}{2} \Rightarrow$ nema realnih nultočaka

3) asimptote. V.A. $x=2$

H.A. $\lim_{x\to\infty} f(x) = \infty \Rightarrow$ nema

K.A. $y = kx + l$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2-5x+7}{x^2-2x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x-5}{2x-2} \overset{L'H}{=} \lim_{x\to\infty} 1 = 1$$

$$\Rightarrow l = \lim_{x\to\infty}(f(x)-x) = \lim_{x\to\infty}\left(\frac{x^2-5x+7}{x-2} - x\right) = \lim_{x\to\infty} \frac{\cancel{x^2}-5x+7-\cancel{x^2}+2x}{x-2} =$$

$$= \lim_{x\to\infty} \frac{7-3x}{x-2} \overset{L'H}{=} \lim_{x\to\infty} \frac{-3}{1} = -3 \Rightarrow \underline{y = x-3} \text{ K.A.}$$

4) $$f'(x) = \frac{(2x-5)(x-2)-(x^2-5x+7)\cdot 1}{(x-2)^2} = \frac{2x^2-9x+10-x^2+5x-7}{(x-2)^2} = \frac{x^2-4x+3}{(x-2)^2} = 0$$

$$\Rightarrow x^2-4x+3=0 \Rightarrow x_{1,2} = \frac{4\pm\sqrt{16-12}}{2} = \frac{4\pm 2}{2} = 2\pm 1 \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 3 \end{matrix}$$

5) $$f''(x) = \frac{(2x-4)(x-2)^{\cancel{2}} - (x^2-4x+3)\cdot 2\cancel{(x-2)}}{(x-2)^{\cancel{4}\,3}} = \frac{\cancel{2x^2}-\cancel{8x}+8-\cancel{2x^2}+\cancel{8x}-6}{(x-2)^3} = \frac{2}{(x-2)^3}$$

$f''(1) = \frac{2}{(1-2)^3} < 0 \Rightarrow (1, f(1)) = \underline{(1,-3)}$ lok. max

$f''(3) = \frac{2}{(3-2)^3} > 0 \Rightarrow (3, f(3)) = \underline{(3,1)}$ lok. min.

6) <u>rast</u>: $f'(x) > 0 \quad \frac{x^2-4x+3}{(x-2)^2} > 0 \Rightarrow x^2-4x+3 > 0 \Rightarrow \boxed{x \in \langle -\infty, 1\rangle \cup \langle 3, \infty\rangle}$

<u>pad</u>: $\underline{x \in \langle 1,3\rangle \setminus \{2\}}$

<u>tok</u>:

	$\langle -\infty,1\rangle$	1	$\langle 1,2\rangle$	2	$\langle 2,3\rangle$	3	$\langle 3,\infty\rangle$
f'	+	0	−		−	0	+
f	↗	MAX	↘		↘	MIN	↗

7) <u>graf</u>: