MATEMATIKA 1

KOLOKVIJI 2004./05. - RJEŠENJA

1. kolokvij
2. kolokvij
3. kolokvij

MATEMATIKA 1 – I. KOLOKVIJ – RJEŠENJA

GRUPA $\boxed{A}$

1. Riješite nejednadžbu $|x-2|-|x^2-5x+6|>0$.

Rješenje. karakteristične točke: $x-2=0 \Rightarrow x=2$

$x^2-5x+6=0 \Rightarrow x=2, x=3$

a) $x\in\langle -\infty, 2]$

$-(x-2)-(x-2)(x-3)>0$

$(x-2)(-1-x+3)>0 \,/\cdot(-1)$

$(x-2)(x-2)<0 \Rightarrow (x-2)^2<0$

ne vrijedi ni za koji $x\in\mathbb{R}$

$\Rightarrow x\in\emptyset$

b) $x\in[2,3]$

$x-2+(x-2)(x-3)>0$

$(x-2)(1+x-3)>0$

$(x-2)(x-2)>0 \Rightarrow (x-2)^2>0$

vrijedi za sve $x\in\mathbb{R}$ osim

za $x=2 \Rightarrow$ rješenje je $x\in\langle 2,3]$

c) $x\in[3,\infty\rangle$

$x-2-(x-2)(x-3)>0$

$(x-2)(1-x+3)>0$

$\underbrace{(x-2)}_{>0 \text{ za sve } x\in[3,\infty\rangle}(-x+4)>0$

$\Rightarrow -x+4>0 \Rightarrow x<4$

$\Rightarrow x\in[3,4\rangle$

konačno rješenje: $\boxed{x\in\langle 2,4\rangle}$

2. način: $|x-2|-|(x-2)(x-3)|>0$

$|x-2|-|x-2|\cdot|x-3|>0$

$\underbrace{|x-2|}_{>0 \text{ za sve } x\in\mathbb{R}}(1-|x-3|)>0 \Rightarrow 1-|x-3|>0 \Rightarrow |x-3|<1 \Rightarrow -1<x-3<1 \,/+3$

$\Rightarrow \boxed{x\in\langle 2,4\rangle}$

2. Ako je $Re(z-1)=0$, izračunajte $(z+1)(\bar{z}+1)-|z|^2$.

Rješenje. $Re(z-1)=0 \Rightarrow Re(z)=1$

$(z+1)(\bar z+1)-|z|^2 = z+\bar z+1+\underbrace{z\bar z-|z|^2}_{=0} = z+\bar z+1 = 2Re(z)+1 = 2\cdot 1+1 = \boxed{3}$

0. jer vrijedi $|z|^2=z\cdot\bar z$

3. U kompleksnoj ravnini prikažite rješenja jednadžbe $(z-1)^6=2^6$.

Rješenje. $(z-1)^6=2^6=2^6(\cos 0+i\sin 0)$

$\Rightarrow z_k-1=\sqrt[6]{2^6}\cdot\left(\cos\left(\frac{0+2k\pi}{6}\right)+i\sin\left(\frac{0+2k\pi}{6}\right)\right)$ $k=0,1,2,3,4,5$

$\Rightarrow z_k=2\left(\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}\right)+1$

$k=0 \Rightarrow z_0=2(\cos 0+i\sin 0)+1=2+1=3$

$k=1 \Rightarrow z_1=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)+1=2\left(\frac12+\frac{\sqrt3}{2}i\right)+1=1+\sqrt3 i+1=2+\sqrt3 i$

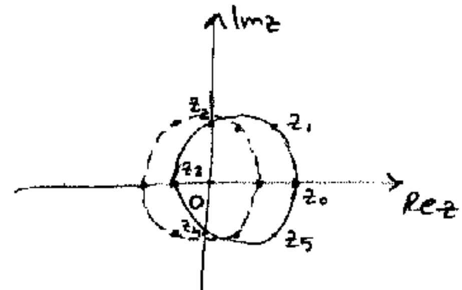
$k=2 \Rightarrow z_2=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)+1=2\left(-\frac12+\frac{\sqrt3}{2}i\right)+1=-1+\sqrt3 i+1=\sqrt3 i$

$k=3 \Rightarrow \quad z_3 = 2(\cos \pi + i \sin \pi) + 1 = 2(-1)+1 = -2+1 = -1$

$k=4 \Rightarrow \quad z_4 = 2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) + 1 = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) + 1 = -1 - \sqrt{3} i + 1 = -\sqrt{3} i$

$k=5 \Rightarrow \quad z_5 = 2\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) + 1 = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) + 1 = 1 - \sqrt{3} i + 1 = 2 - \sqrt{3} i$

Prikaz u kompleksnoj ravnini:



Sve točke su na kružnici radijus 2 sa središtem u $z=1$!

4. Koristeći matematičku indukciju dokažite da je $13^{2n}+6$ djeljivo sa 7 za svaki nenegativni cijeli broj.

Rješenje.

1) baza: $n=0 \quad 13^{2\cdot 0}+6 = 1+6 = 7$ ✓

2) pretpostavka: pretpostavljamo da tvrdnja vrijedi za neki $n \in \mathbb{N}_0$, tj. da je $13^{2n}+6$ djeljivo sa $7 \Rightarrow \exists M \in \mathbb{N}$ takav da je $13^{2n}+6 = 7N$

3) korak: pomoću pretpostavke dokazujemo da tvrdnja vrijedi za $n+1$, tj. da je $13^{2(n+1)}+6$ djeljiv sa $7 \Rightarrow \exists N \in \mathbb{N}$ takav da je $13^{2n+2}+6 = 7M$.

Dokaz. $13^{2n+2}+6 = 13^2 \cdot 13^{2n} + 6 = 169 \cdot 13^{2n} + 6 \overset{PP}{=} 169(7N-6)+6 =$
$= 169 \cdot 7N - 6 \cdot 169 + 6 = 169 \cdot 7N - 6 \cdot 168 = 7(\underbrace{169N - 6 \cdot 24}_{M}) = 7M$ ✓

4) zaključak: tvrdnja vrijedi za svaki $n \in \mathbb{N}_0$.

5. Ako je x_0 algebarski broj, dokažite da je tada i $\sqrt{x_0}$ algebarski.

Rješenje. x_0 je algebarski $\Rightarrow \exists\, f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_n \neq 0$, $a_0, \dots, a_n \in \mathbb{Z}$ i $f(x_0) = 0$

$$\Rightarrow \quad a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_1 x_0 + a_0 = 0 \qquad (*)$$

Definiramo $g(x) := a_n x^{2n} + a_{n-1} x^{2n-2} + \dots + a_1 x^2 + a_0$

Sada je $g(\sqrt{x_0}) = a_n \sqrt{x_0}^{2n} + a_{n-1} \sqrt{x_0}^{2n-2} + \dots + a_1 \sqrt{x_0}^2 + a_0 =$
$= a_n (\sqrt{x_0}^2)^n + a_{n-1} (\sqrt{x_0}^2)^{n-1} + \dots + a_1 \sqrt{x_0}^2 + a_0 =$
$= a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_1 x_0 + a_0 = 0$ zbog $(*)$, pa je

$g(x)$ traženi polinom za $\sqrt{x_0}$ i ima sva zahtijevana svojstva

$\Rightarrow \sqrt{x_0}$ je algebarski, što je i trebalo dokazati.

MATEMATIKA 1 – 2 KOLOKVIJ, GRUPA B

1. $$\begin{aligned} x-y-z&=-1\\ 2x-y+z&=-5\\ x+y-z&=5 \end{aligned} \qquad p\ldots\ \frac{x-2}{1}=\frac{y-4}{2}=\frac{z-6}{3}$$

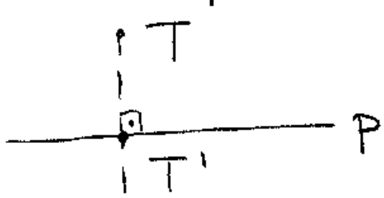
Rj.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1\\ 2 & -1 & 1 & -5\\ 1 & 1 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1\\ 0 & 1 & 3 & -3\\ 0 & 2 & 0 & 6 \end{array}\right) \Rightarrow$$

$$2y=6 \Rightarrow y=3$$
$$y+3z=-3$$
$$3z=-3-3=-6$$
$$z=-2$$
$$x-y-z=-1$$
$$\Rightarrow x=-1+3-2=0$$

$$T(0,3,-2)$$

Tražimo projekciju T' od T na pravac p:



$\pi\ldots$ ravnina koja sadrži točku T i $\perp$ je na pravac p

$$\vec{n}_\pi=\vec{s}_p=\vec{i}+2\vec{j}+3\vec{k}$$

$$\Rightarrow \pi\ldots\ (x-0)+2(y-3)+3(z+2)=0$$

$T'=$ presjek od p i π

$$p\ldots\ \left.\begin{aligned} x&=2+\lambda\\ y&=4+2\lambda\\ z&=6+3\lambda \end{aligned}\right\} \Rightarrow 2+\underline{\lambda}+2(4+\underline{2\lambda}-3)+3(6+\underline{3\lambda}+2)=0$$

$$14\lambda=-2-2-24=-28$$
$$\lambda=-2$$

$$\Rightarrow T'(0,0,0)$$

$$d(T,T')=\sqrt{0^2+3^2+2^2}=\sqrt{9+4}=\sqrt{13}$$

2. Nađite $\lambda\in\mathbb{R}$ t.d. sustav
$$\begin{aligned} x+\lambda y-2z&=0\\ y-2\lambda z&=0\\ x+\lambda y-2\lambda z&=0 \end{aligned}$$
ima besk. rj. i geom. int. rješenje.

Rj.

$$\left(\begin{array}{ccc|c} 1 & \lambda & -2 & 0\\ 0 & 1 & -2\lambda & 0\\ 1 & \lambda & -2\lambda & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \lambda & -2 & 0\\ 0 & 1 & -2\lambda & 0\\ 0 & 0 & -2\lambda+2 & 0 \end{array}\right)$$

$$\Rightarrow (-2\lambda+2)z=0 \quad \text{tj.} \quad (1-\lambda)z=0$$

a) $1-\lambda\neq 0 \Rightarrow z=0$
$$\left.\begin{aligned} &y-2\lambda z=0 \Rightarrow y=0\\ &\text{i isto tako } x=0 \end{aligned}\right\}\ \text{rj. je jedinstv}$$

b) $1-\lambda=0 \Rightarrow \lambda=1$ pa sustav izgleda

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \quad y-2z=0, \quad z=t$$

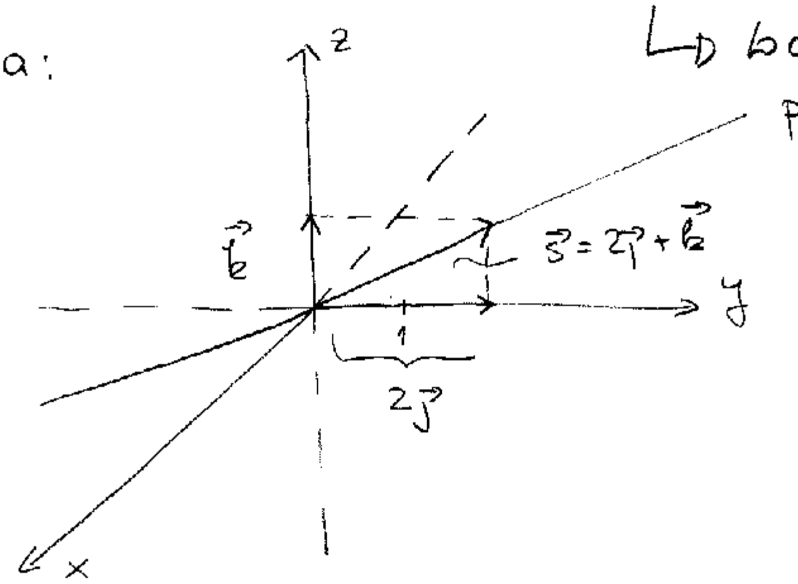
$$y=2t$$

$$x+y-2z=0 \Rightarrow x=0$$

Rješenje je pravac $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

↳ točka ↳ vektor smjera

skica:



3. Izračunati $(AB)^{-1}$ ako je $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

Rj $\quad (AB)^{-1} = B^{-1}A^{-1}$

B^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{matrix}}_{B^{-1}}$$

A^{-1}:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

$$B^{-1}A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

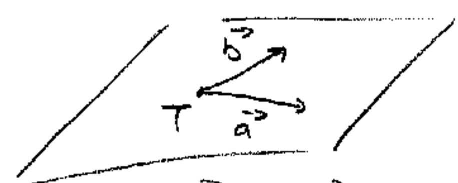
4. $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{c} = 5\vec{i} + 4\vec{j} - \vec{k}$

$T(1,1,1)$

Rj $\quad \vec{c} = \lambda \vec{a} + \mu \vec{b}$

$$\Rightarrow \left.\begin{aligned} 5 &= \lambda + \mu \\ 4 &= -\lambda + 2\mu \\ -1 &= \lambda - \mu \end{aligned}\right\} + \Rightarrow 4 = 2\lambda \Rightarrow \lambda = 2, \quad \mu = 3$$

$$\vec{c} = 2\vec{a} + 3\vec{b}$$



$$\vec{n}_\pi = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}(+1-2) - \vec{j}(-1-1) + \vec{k}(2+1) = -\vec{i} + 2\vec{j} + 3\vec{k}$$

$\pi \ldots -(x-1) + 2(y-1) + 3(z-1)$

5. A inv., realna i $A^6 = A$, $\det A = ?$

Rj $\quad A$ inv $\Rightarrow \det A \neq 0$

A realna $\Rightarrow \det A \in \mathbb{R}$

$\det(A \cdot A) = (\det A)^2$

Indukcijom se lako vidi $\det A^n = (\det A)^n$

$\Rightarrow \det A^6 = \det A$ iz $A^6 = A$

$\Rightarrow (\det A)^6 = \det A$, $\det A = t$

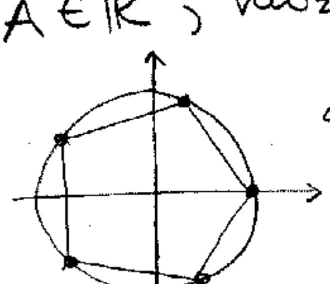
Rješavamo jdu. $t^6 = t$

$\Rightarrow t(t^5 - 1) = 0 \Rightarrow t = 0$ ili $t^5 = 1$

$\det A = 0$ ne može jer je A regularna.

$\Rightarrow (\det A)^5 = 1$

Jer $\det A \in \mathbb{R}$, nužno slijedi $\det A = 1$



rješenja od $t^5 = 1$ čine pravilan peterokut i jedino realno je $t = 1$, vidi se već iz skice

3. KOLOKVIJ IZ MATEMATIKE 1, 2. veljače 2005. (B)

Pokažite da se hiperbole $xy=2$ i $x^2-y^2=3$ sijeku pod pravim kutem.

Rj. Neka se hiperbole sijeku u točki $T=(x(T),y(T))$

$$\Rightarrow y(T)=\frac{2}{x(T)}$$

Gledamo koeficijente smjera tangenti. To su derivacije u T:

$$xy=2 \Rightarrow y' = -\frac{2}{x^2} \Rightarrow y_1'(T) = -\frac{2}{(x(T))^2}$$

$$x^2-y^2=3 \Rightarrow 2x-2y\cdot y'=0 \Rightarrow y_2'(T)=\frac{x(T)}{y(T)}$$

Jer je to presjecište, $y(T)=\frac{2}{x(T)}$

$$\Rightarrow y_2'(T)=\frac{x(T)}{\frac{2}{x(T)}}=\frac{x^2(T)}{2}$$

$$\Rightarrow y_1'(T)y_2'(T)=-\frac{2}{(x(T))^2}\cdot\frac{(x(T))^2}{2}=-1$$

pa su tangente okomite $\Rightarrow$ y_1 i y_2 sijeku se pod pravim $\sphericalangle$

Za koje λ je domena od f čitav $\mathbb{R}$ ako je f zadana sa $f(x)=\sqrt{\lambda-\cos x}+\ln(\lambda(2+|x|))$?

Rj. zbog $\sqrt{}$: $\lambda-\cos x \geq 0 \Rightarrow \lambda \geq \cos x$

Jer $\cos x \in [-1,1] \Rightarrow$ dovoljno je staviti $\lambda \geq 1$

zbog $\ln$: $\lambda(2+|x|)>0$. Jer $2+|x|>0 \Rightarrow \lambda>0$

Stoga je dovoljno uzeti $\lambda\in[1,+\infty)$, $D(f)=\mathbb{R}$

Nacrtajte graf funkcije $f(x)=\frac{8}{x^2-4}$

Rj.

1) $D(f)=\mathbb{R}\setminus\{\pm 2\}$

2) $N(f)=\emptyset$

3) asimptote: V. A. $x=2$, $x=-2$

H. A. $\lim\limits_{x\to\pm\infty}\frac{8}{x^2-4}=0$

$\Rightarrow$ kosih asimptota nema.

4) kandidati za ekstreme:

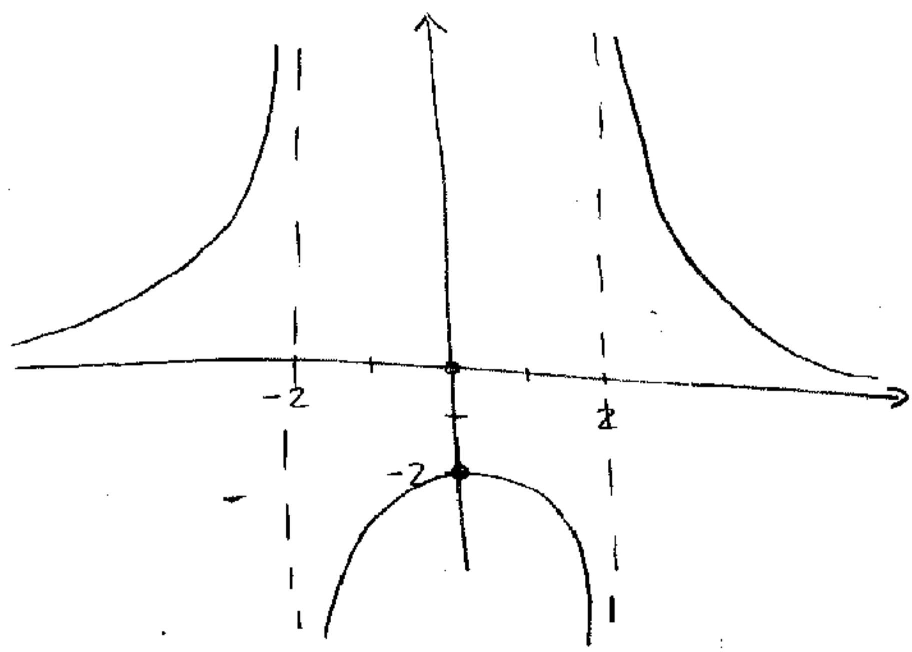
$$f'(x) = -\frac{2x}{(x^2-4)^2} \Rightarrow x = 0 \quad , \quad f(0) = -2$$

$$f''(x) = -\frac{2(x^2-4)^2 - 2x\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4} \Rightarrow f''(0) = -2$$

pa je tu lok. mak.

5) tok: $f'(x) = -\dfrac{2x}{(x^2-4)^2}$

pa $f'(x) > 0$ za $x < 0$ funkcija raste
$f'(x) < 0$ za $x > 0$ funkcija pada



4. Izračunajte bez upotrebe L'H pravila limes

$$\lim_{x\to 3} \frac{9-x^2}{\sin\frac{\pi x}{3}}$$

Rj

$$\lim_{x\to 3} \frac{9-x^2}{\sin\frac{\pi x}{3}} = \left\{ \begin{array}{l} t = x-3 \\ t\to 0 \text{ kada } x\to 3 \end{array} \right\} = \lim_{t\to 0} \frac{9-(t+3)^2}{\sin\frac{\pi}{3}(t+3)} =$$

$$= \lim_{t\to 0} \frac{\cancel{9}-t^2-6t-\cancel{9}}{\sin(\frac{\pi}{3}t+\pi)} = \lim_{t\to 0} \frac{-t^2-6t}{-\sin\frac{\pi}{3}t} = \lim_{t\to 0} \frac{t+6}{\underbrace{\frac{\sin\frac{\pi}{3}t}{\frac{\pi}{3}t}}_{\to 1}\cdot\frac{\pi}{3}} = \frac{18}{\pi}$$

5. Broj 3 rastavite na dva pribrojnika tako da produkt bude najveći.

Rj $3 = x+y \Rightarrow y = 3-x$, $f(x) = xy = x(3-x) = 3x-x^2$
$f'(x) = 3-2x$ pa $f'(x) = 0 \Rightarrow x = \frac{3}{2}$, $f''(x) = -2$ pa je tu lok. max. $\Rightarrow x = \frac{3}{2}$ $y = \frac{3}{2}$ je rješenje.