

MATEMATIKA 1

KOLOKVIJI 2004./05. - RJEŠENJA

1. kolokvij
2. kolokvij
3. kolokvij

1. Riješite nejednadžbu $|x-2| - |x^2-5x+6| > 0$.

Rješenje: karakteristične točke: $x-2=0 \Rightarrow x=2$

$$x^2-5x+6=0 \Rightarrow x=2, x=3$$

a) $x \in (-\infty, 2]$

$$-(x-2) - (x-2)(x-3) > 0$$

$$(x-2)(-1-x+3) > 0 \quad | \cdot (-1)$$

$$(x-2)(x-2) < 0 \Rightarrow (x-2)^2 < 0$$

ne vrijedi ni za koji $x \in \mathbb{R}$

$$\Rightarrow x \in \emptyset$$

b) $x \in [2, 3]$

$$x-2 + (x-2)(x-3) > 0$$

$$(x-2)(1+x-3) > 0$$

$$(x-2)(x-2) > 0 \Rightarrow (x-2)^2 > 0$$

vrijedi za sve $x \in \mathbb{R}$ osim

$$\text{za } x=2 \Rightarrow \text{rješenje je } x \in \langle 2, 3 \rangle$$

c) $x \in [3, \infty)$

$$x-2 - (x-2)(x-3) > 0$$

$$(x-2)(1-x+3) > 0$$

$$(x-2)(-x+4) > 0$$

\forall za sve $x \in [3, \infty)$

$$\Rightarrow -x+4 > 0 \Rightarrow x < 4$$

$$\Rightarrow x \in [3, 4)$$

konačno rješenje: $x \in \langle 2, 4 \rangle$

2. način: $|x-2| - |(x-2)(x-3)| > 0$

$$|x-2| - |x-2| \cdot |x-3| > 0$$

$$|x-2|(1-|x-3|) > 0$$

$$\forall \text{ za sve } x \in \mathbb{R} \Rightarrow 1-|x-3| > 0 \Rightarrow |x-3| < 1 \Rightarrow -1 < x-3 < 1 \quad | +3$$

$$\Rightarrow x \in \langle 2, 4 \rangle$$

2. Ako je $\operatorname{Re}(z-1)=0$, izračunajte $(z+1)(\bar{z}+1) - |z|^2$.

Rješenje: $\operatorname{Re}(z-1)=0 \Rightarrow \operatorname{Re}(z)=1$

$$(z+1)(\bar{z}+1) - |z|^2 = z + \bar{z} + 1 + \underbrace{z\bar{z}}_{|z|^2} - |z|^2 = z + \bar{z} + 1 = 2\operatorname{Re}(z) + 1 = 2 \cdot 1 + 1 = \boxed{3}$$

|| jer vrijedi $|z|^2 = z\bar{z}$

3. U kompleksnoj ravni prikazite rješenja jednadžbe $(z-1)^6 = 2^6$.

Rješenje: $(z-1)^6 = 2^6 = 2^6(\cos 0 + i \sin 0)$

$$\rightarrow z_k - 1 = \sqrt[6]{2^6} \cdot \left(\cos\left(\frac{0+2k\pi}{6}\right) + i \sin\left(\frac{0+2k\pi}{6}\right) \right) \quad k=0,1,2,3,4,5$$

$$\Rightarrow z_k = 2 \left(\cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3} \right) + 1$$

$$k=0 \Rightarrow z_0 = 2(\cos 0 + i \sin 0) + 1 = 2 + 1 = 3$$

$$k=1 \Rightarrow z_1 = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) + 1 = 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) + 1 = 1 + \sqrt{3}i + 1 = 2 + \sqrt{3}i$$

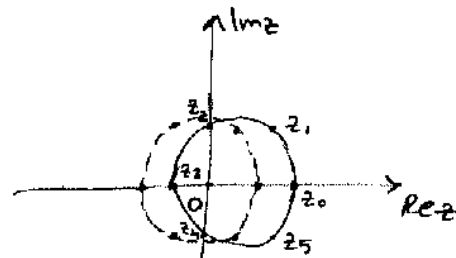
$$k=2 \Rightarrow z_2 = 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) + 1 = 2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) + 1 = -1 + \sqrt{3}i + 1 = \sqrt{3}i$$

$$k=3 \Rightarrow z_3 = 2(\cos \pi + i \sin \pi) + 1 = 2(-1) + 1 = -2 + 1 = -1$$

$$k=4 \Rightarrow z_4 = 2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) + 1 = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 1 = -1 - \sqrt{3}i + 1 = -\sqrt{3}i$$

$$k=5 \Rightarrow z_5 = 2\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) + 1 = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 1 = 1 - \sqrt{3}i + 1 = 2 - \sqrt{3}i$$

Prikaz u kompleksnoj ravni:



Sve tačke su na krivici radijus
2 sa središtem u $z=1$!

4. Koristeći matematičku indukciju
dokažite da je $13^{2n} + 6$ djeljivo sa
7 za svaki nenegativni cijeli broj.

Rješenje.

1) baza: $n=0$ $13^{2 \cdot 0} + 6 = 1 + 6 = 7 \checkmark$

2) pretpostavka: pretpostavljamo da tvrdnja vrijedi za neki $n \in \mathbb{N}_0$, tj.
da je $13^{2n} + 6$ djeljivo sa 7 $\Rightarrow \exists M \in \mathbb{N}$ takav da
je $13^{2n} + 6 = 7M$.

3) korak: pomoću pretpostavke dokažujemo da tvrdnja vrijedi za $n+1$, tj. da
je $13^{2(n+1)} + 6$ djeljivo sa 7 $\Rightarrow \exists N \in \mathbb{N}$ takav da je
 $13^{2n+2} + 6 = 7M$.

Dokaz. $13^{2n+2} + 6 = 13^2 \cdot 13^{2n} + 6 = 169 \cdot 13^{2n} + 6 \stackrel{pp}{=} 169(7M - 6) + 6 =$
 $= 169 \cdot 7M - 6 \cdot 169 + 6 = 169 \cdot 7M - 6 \cdot 168 = 7(169M - 6 \cdot 24) = 7M \checkmark$

4) zaključak: tvrdnja vrijedi za svaki $n \in \mathbb{N}_0$.

5. Ako je x_0 algebarski broj, dokažite da je tada i $\sqrt{x_0}$ algebarski.

Rješenje. x_0 je algebarski $\Rightarrow \exists f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_n \neq 0$,
 $a_0, \dots, a_n \in \mathbb{Z}$ i $f(x_0) = 0$

$$\Rightarrow a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_1 x_0 + a_0 = 0 \quad (*)$$

Definiramo $g(x) := a_n x^{2n} + a_{n-1} x^{2n-2} + \dots + a_1 x^2 + a_0$

$$\begin{aligned} \text{Sada je } g(\sqrt{x_0}) &= a_n \sqrt{x_0}^{2n} + a_{n-1} \sqrt{x_0}^{2n-2} + \dots + a_1 \sqrt{x_0}^2 + a_0 = \\ &= a_n (\sqrt{x_0}^2)^n + a_{n-1} (\sqrt{x_0}^2)^{n-1} + \dots + a_1 \sqrt{x_0}^2 + a_0 = \\ &= a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_1 x_0 + a_0 = 0 \text{ zbog } (*), \text{ pa je} \end{aligned}$$

$g(x)$ traženi polinom za $\sqrt{x_0}$ i ima sva zahtijevana svojstva

$\Rightarrow \sqrt{x_0}$ je algebarski, što je i trebalo dokazati.

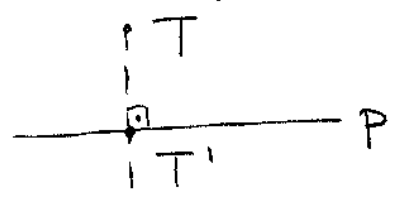
1.
$$\begin{aligned} x - y - z &= -1 \\ 2x - y + z &= -5 \\ x + y - z &= 5 \end{aligned}$$

$$p \dots \frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{3}$$

Rj:
$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ 2 & -1 & 1 & -5 \\ 1 & 1 & -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 2 & 0 & 6 \end{pmatrix} \Rightarrow \begin{aligned} 2y &= 6 \Rightarrow y = 3 \\ y + 3z &= -3 \\ 3z &= -3 - 3 = -6 \\ z &= -2 \\ x - y - z &= -1 \end{aligned}$$

$T(0, 3, -2) \Rightarrow x = -1 + 3 - 2 = 0$

Tražimo projekciju T' od T na pravac p :
 $\pi \dots$ ravnina koja sadrži točku T i \perp je na pravcu p .



$\vec{n}_\pi = \vec{s}_p = 2\vec{i} + 2\vec{j} + 3\vec{k}$

$\Rightarrow \pi \dots (x-0) + 2(y-3) + 3(z+2) = 0$

$T' =$ presjek od p i π

$$\begin{aligned} p \dots \begin{cases} x = 2 + \lambda \\ y = 4 + 2\lambda \\ z = 6 + 3\lambda \end{cases} \Rightarrow \begin{aligned} 2 + \lambda + 2(4 + 2\lambda - 3) + 3(6 + 3\lambda + 2) &= 0 \\ 14\lambda &= -2 - 2 - 24 = -28 \\ \lambda &= -2 \end{aligned} \end{aligned}$$

$\Rightarrow T'(0, 0, 0)$

$d(T, T') = \sqrt{0^2 + 3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$

2. Nađite $\lambda \in \mathbb{R}$ t.d. sustav $\begin{cases} x + \lambda y - 2z = 0 \\ y - 2\lambda z = 0 \\ x + \lambda y - 2\lambda z = 0 \end{cases}$ ima besk. rj. i geom. int. rješenje.

Rj:
$$\begin{pmatrix} 1 & \lambda & -2 & 0 \\ 0 & 1 & -2\lambda & 0 \\ 1 & \lambda & -2\lambda & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & -2 & 0 \\ 0 & 1 & -2\lambda & 0 \\ 0 & 0 & -2\lambda + 2 & 0 \end{pmatrix}$$

$\Rightarrow (-2\lambda + 2)z = 0$ tj. $(1 - \lambda)z = 0$

a) $1 - \lambda \neq 0 \Rightarrow z = 0$
 $y - 2\lambda z = 0 \Rightarrow y = 0$
 i isto tako $x = 0$ } rj. je jedinstveno

f) $1-\lambda=0 \Rightarrow \lambda=1$ pa sustav izgleda

$$\begin{pmatrix} 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow$$

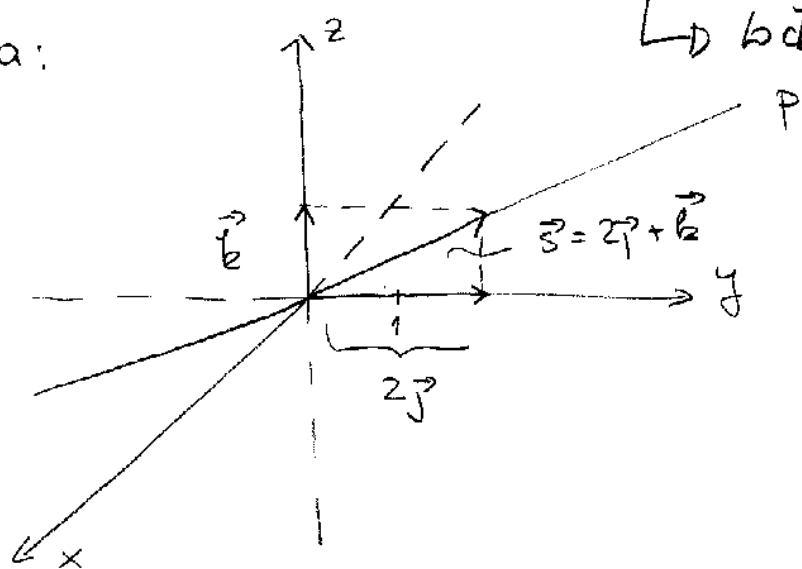
$$y-2z=0, \quad z=t$$

$$y=2t$$

$$x+y-2z=0 \Rightarrow x=0$$

Rešenje je pravac $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

skica:



\hookrightarrow bodža \hookrightarrow vektor smjera

3. Izračunati $(AB)^{-1}$ ako je $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

B^{-1} :

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ -1 & 0 & 0 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & -1 & | & 1 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & -1 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & 1 & -1 \end{pmatrix}$$

A^{-1} :

$$\begin{pmatrix} 0 & 0 & -1 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 0 & -1 & | & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & -1 & 1 & -1 \\ 0 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & 0 & 0 \end{pmatrix}$$

$B^{-1}A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

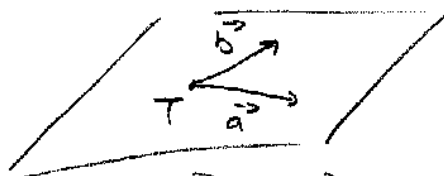
4. $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{c} = 5\vec{i} + 4\vec{j} - \vec{k}$
 $T(1, 1, 1)$

Rj $\vec{c} = \lambda \vec{a} + \mu \vec{b}$

$$\Rightarrow \begin{cases} 5 = \lambda + \mu \\ 4 = -\lambda + 2\mu \\ -1 = \lambda - \mu \end{cases} \Rightarrow 4 = 2\lambda \Rightarrow \lambda = 2$$

$$\mu = 3 \quad \checkmark$$

$$\vec{c} = 2\vec{a} + 3\vec{b}$$



$$\vec{n}_\pi = \vec{a} \times \vec{b} =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}(+1-2) - \vec{j}(-1-1) + \vec{k}(2+1) = -\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\pi \dots - (x-1) + 2(y-1) + 3(z-1)$$

5. A inv., realna i $A^6 = A$, $\det A = ?$

Rj A inv $\Rightarrow \det A \neq 0$

A realna $\Rightarrow \det A \in \mathbb{R}$

$$\det(A \cdot A) = (\det A)^2$$

Indukcijom se lako vidi $\det A^n = (\det A)^n$

$$\Rightarrow \det A^6 = \det A \text{ iz } A^6 = A$$

$$\Rightarrow (\det A)^6 = \det A, \det A = t$$

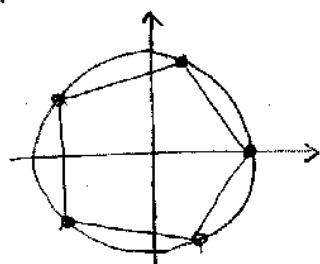
Rješavamo jedn. $t^6 = t$

$$\Rightarrow t(t^5 - 1) = 0 \Rightarrow t = 0 \text{ ili } t^5 = 1$$

$\det A = 0$ ne može jer je A regularna.

$$\Rightarrow (\det A)^5 = 1$$

Zer $\det A \in \mathbb{R}$, nužno slijedi $\det A = 1$



← rješenja od $t^5 = 1$ čine pramčan peterokut i jednino realno je $t = 1$, vidi se već iz skice

Pokažite da se hiperbole $xy=2$ i $x^2-y^2=3$ sijeku pod pravim kutem.

Rj. Neka se hiperbole sijeku u točki $T=(x(T), y(T))$

$$\Rightarrow y(T) = \frac{2}{x(T)}$$

Gledamo koeficijente smjera tangenti. To su derivacije u T :

$$xy=2 \Rightarrow y' = -\frac{2}{x^2} \Rightarrow y_1'(T) = -\frac{2}{(x(T))^2}$$

$$x^2-y^2=3 \Rightarrow 2x-2y \cdot y' = 0 \Rightarrow y_2'(T) = \frac{x(T)}{y(T)}$$

Jer je to presjeciste, $y(T) = \frac{2}{x(T)}$

$$\Rightarrow y_2'(T) = \frac{x(T)}{\frac{2}{x(T)}} = \frac{x^2(T)}{2}$$

$$\Rightarrow y_1'(T) y_2'(T) = -\frac{2}{(x(T))^2} \cdot \frac{(x(T))^2}{2} = -1$$

pa su tangente okomite $\Rightarrow y_1$ i y_2 sijeku se pod pravim kutem.

Za koje λ je domena od f čitav \mathbb{R} ako je f zadana sa

$$f(x) = \sqrt{\lambda - \cos x} + \ln(\lambda(2+|x|))?$$

Rj. zbog $\sqrt{\quad}$: $\lambda - \cos x \geq 0 \Rightarrow \lambda \geq \cos x$

Jer $\cos x \in [-1, 1] \Rightarrow$ dovoljno je staviti $\lambda \geq 1$

zbog \ln : $\lambda(2+|x|) > 0$. Jer $2+|x| > 0 \Rightarrow \lambda > 0$

Svega je dovoljno uzeti $\lambda \in [1, +\infty)$; $D(f) = \mathbb{R}$

Nacrtajte graf funkcije $f(x) = \frac{8}{x^2-4}$

- Rj.
- $D(f) = \mathbb{R} \setminus \{\pm 2\}$
 - $N(f) = \emptyset$
 - asimptote: V. A. $x=2, x=-2$
H. A. $\lim_{x \rightarrow \pm\infty} \frac{8}{x^2-4} = 0$

\Rightarrow kosih asimptota nema.

4) kandidati za ekstreme:

$$f'(x) = -\frac{2x}{(x^2-4)^2} \Rightarrow x=0, \quad f(0) = -2$$

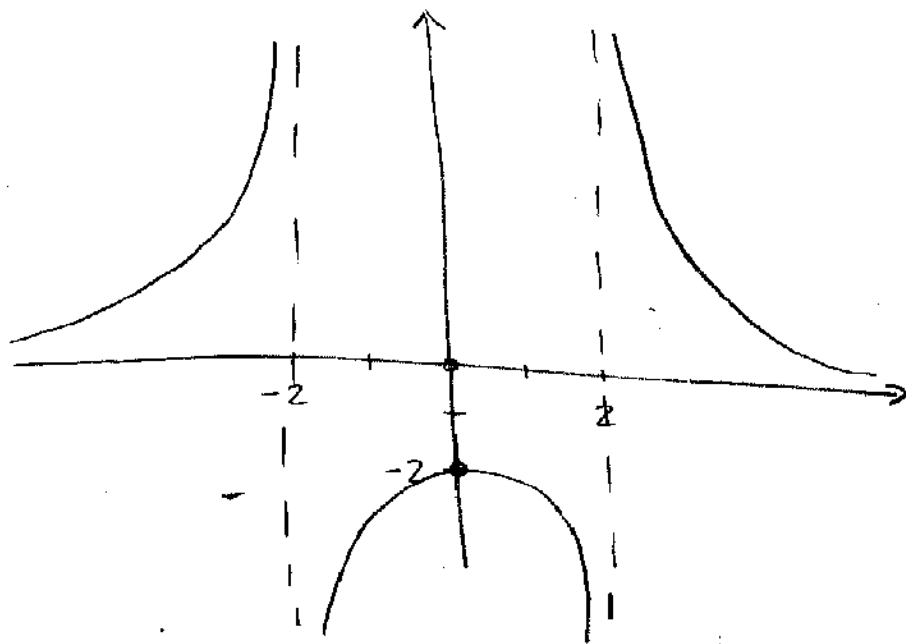
$$f''(x) = -\frac{2(x^2-4)^2 - 2x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} \Rightarrow f''(0) = -2$$

pa je tu lok. max.

5) točk: $f'(x) = -\frac{2x}{(x^2-4)^2}$

pa $f'(x) > 0$ za $x < 0$
 $f'(x) < 0$ za $x > 0$

funkcija raste
 funkcija pada



4. Izračunajte bez upotrebe L'H pravila limes

$$\lim_{x \rightarrow 3} \frac{9-x^2}{\sin \frac{\pi x}{3}}$$

R_j $\lim_{x \rightarrow 3} \frac{9-x^2}{\sin \frac{\pi x}{3}} = \left\{ \begin{array}{l} t = x-3 \\ t \rightarrow 0 \text{ kada } x \rightarrow 3 \end{array} \right\} = \lim_{t \rightarrow 0} \frac{9-(t+3)^2}{\sin \frac{\pi}{3}(t+3)} =$

$$= \lim_{t \rightarrow 0} \frac{9-t^2-6t-9}{\sin(\frac{\pi}{3}t + \pi)} = \lim_{t \rightarrow 0} \frac{-t^2-6t}{-\sin \frac{\pi}{3}t} = \lim_{t \rightarrow 0} \frac{t+6}{\frac{\sin \frac{\pi}{3}t}{\frac{\pi}{3}t} \cdot \frac{\pi}{3}} = \frac{18}{\pi}$$

(Note: $\frac{\sin \frac{\pi}{3}t}{\frac{\pi}{3}t} \rightarrow 1$)

5. Broj 3 rastavite na dva prirodnika tako da produkt bude najveći.

R_j $3 = x+y \Rightarrow y = 3-x, \quad f(x) = xy = x(3-x) = 3x-x^2$
 $f'(x) = 3-2x$ pa $f'(x) = 0 \Rightarrow x = \frac{3}{2}, \quad f''(x) = -2$ pa je tu lok. max. $\Rightarrow x = \frac{3}{2} \quad y = \frac{3}{2}$ je rešenje.