MATEMATIKA 1

PISMENI ISPITI 2004. - RJEŠENJA

15. svibnja
23. lipnja
6. srpnja
13. srpnja
9. rujna
23. rujna
1. listopada
11. prosinca

MATEMATIKA 1 – PISMENI ISPIT – 15.5.04.

Zadatak. Odredite sve prirodne brojeve a takve da je domena funkcije $f(x)=\log_a(x^2+ax+3)$ čitav skup $\mathbb{R}$.

Rješenje. uvjeti: $a>0, a\neq 1$ (zbog baze log. funkcije)

$x^2+ax+3>0$ (zbog argumenta log. funkcije)

Želimo da su rješenja ovih uvjeta svi $x\in\mathbb{R}$.

Da bi $x^2+ax+3>0$ bio cijeli $\mathbb{R}$, mora vrijediti $g(x)>0$ za sve $x\in\mathbb{R}$, gdje je $g(x)=x^2+ax+3$, a to će biti zadovoljeno ako i samo ako jednadžba $g(x)=0$ nema realnih rješenja, tj. ako i samo ako je $D=a^2-4\cdot3\cdot1=a^2-12<0 \Rightarrow a^2<12 \Rightarrow |a|<2\sqrt{3}$. Sada iz prvog uvjeta $(a>0, a\neq1)$ slijedi $\boxed{a\in\{2,3\}}$.

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte $\lim\limits_{x\to-2}\frac{\mathrm{tg}(\pi x)}{x+2}$.

Rješenje. supst. $x+2=t \Rightarrow t\to0$

$$\lim_{t\to0}\frac{\mathrm{tg}\,\pi(t-2)}{t}=\lim_{t\to0}\frac{\sin(\pi t-2\pi)}{t\cdot\cos(\pi t-2\pi)}=$$

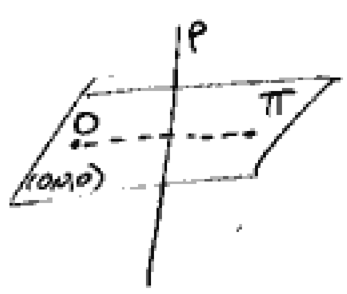
$$=\lim_{t\to0}\frac{\sin\pi t\cdot\overbrace{\cos2\pi}^{=1}-\overbrace{\cancel{\cos\pi t\cdot\sin2\pi}}^{=0}}{t(\cos\pi t\cdot\underbrace{\cos2\pi}_{=1}+\cancel{\sin\pi t\cdot\sin2\pi})}=$$

$$=\lim_{t\to0}\frac{\sin\pi t}{t\cdot\cos\pi t}=\lim_{t\to0}\underbrace{\frac{\sin\pi t}{\pi t}}_{1}\cdot\pi\cdot\underbrace{\frac{1}{\cos\pi t}}_{1}=\boxed{\pi}.$$

Zadatak. Odredite točku simetričnu ishodištu s obzirom na pravac dan jednadžbom $\frac{x-5}{1}=\frac{y}{2}=\frac{z+3}{-3}$.

Rješenje. Položimo ravninu π kroz $(0,0,0)$ okomitu na pravac $\Rightarrow$

$\vec{n}=\vec{i}+2\vec{j}-3\vec{k} \Rightarrow$ jednadžba ravnine je $x+2y-3z=0$.



Presjek pravca i ravnine tražimo iz parametarskog oblika jednadžbe pravca $x=t+5$, $y=2t$, $z=-3t-3$ uvrštavanjem u jednadžbu ravnine:

$$t+5+4t+9t+9=0$$

$14t=-14 \Rightarrow t=-1 \Rightarrow$ dobili smo točku O';

$O'(4,-2,0)$.

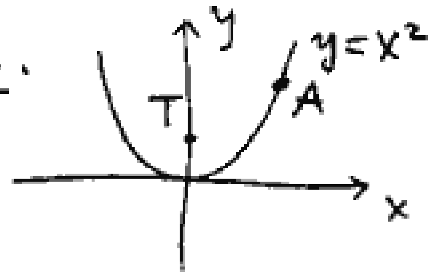
Da bismo našli točku O'' simetričnu $O(0,0,0)$, koristimo identitet

$\overrightarrow{OO''} = 2\overrightarrow{OO'}$, gdje $O'' = (x,y,z)$:

$\Rightarrow x\vec{i} + y\vec{j} + z\vec{k} = 2(4\vec{i} - 2\vec{j}) = 8\vec{i} - 4\vec{j} \Rightarrow \left.\begin{matrix} x = 8 \\ y = -4 \\ z = 0 \end{matrix}\right\} \Rightarrow \boxed{O'' = (8,-4,0)}$

Zadatak. Na krivulji $y = x^2$ nađite točku najbližu točki $T(0,1)$.

Rješenje.



Uzmimo $A = (x, x^2)$ s krivulje i računamo udaljenost $d = |\overline{AT}| =$

$= \sqrt{(x-0)^2 + (x^2-1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} =$

$= \sqrt{x^4 - x^2 + 1}$.

Minimum udaljenosti se postiže kad funkcija $f(x) = x^4 - x^2 + 1$ postiže minimum.

$f'(x) = 4x^3 - 2x = 0$

$2x(2x^2-1) = 0 \Rightarrow \quad x_1 = 0,\ x_2 = \frac{\sqrt{2}}{2},\ x_3 = -\frac{\sqrt{2}}{2}$

$f''(x) = 12x^2 - 2$

$f''(0) = -2 < 0$ max

$f''(\pm\frac{\sqrt{2}}{2}) = 12 \cdot \frac{1}{2} - 2 = 4 > 0$ min

$\Rightarrow$ imamo dva rješenja: $A_1(\frac{\sqrt{2}}{2}, \frac{1}{2})$ i $A_2(-\frac{\sqrt{2}}{2}, \frac{1}{2})$. Minimalna udaljenost iznosi $d_{min} = \frac{\sqrt{3}}{2}$.

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \dfrac{x^2 - x}{(x+1)^2}$.

Rješenje.

1) $N(f) = \{0, 1\}$ $\quad x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x_1 = 0,\ x_2 = 1$

2) $D(f) = \mathbb{R} \setminus \{-1\}$

3) asimptote:

V.A. $x = -1$

H.A. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2 - x}{x^2 + 2x + 1} = 1$

$\Rightarrow y = 1$

K.A. nema

4) $f'(x) = \dfrac{(2x-1)(x+1)^2 - (x^2-x)\cdot 2(x+1)}{(x+1)^4} =$

$$= \frac{(\cancel{x+1})(3\cancel{x^2}+2x-x-1-2\cancel{x^2}+2x)}{(x+1)^{\cancel{4}}} = \frac{3x-1}{(x+1)^3} = 0 \Rightarrow \underline{x=\frac{1}{3}}$$

kandidat za ekstrem

$$f''(x) = \frac{3(x+1)^3-(3x-1)\cdot 3(x+1)^2}{(x+1)^6} = \frac{3\cancel{(x+1)^2}(x+1-3x+1)}{(x+1)^{\cancel{6}\,4}} = \frac{6-6x}{(x+1)^4}$$

$$f''\left(\frac{1}{3}\right) = \frac{6-\cancel{6}\frac{1}{\cancel{3}}}{\left(\frac{1}{3}+1\right)^4} > 0 \Rightarrow$$ minimum; iznosi $$f\left(\frac{1}{3}\right) = \frac{\frac{1}{9}-\frac{1}{3}}{\left(\frac{1}{3}+1\right)^2} = \underline{\underline{-\frac{1}{8}}}$$

5) $$f'(x) > 0 \Rightarrow \frac{3x-1}{(x+1)^3} = \frac{3x-1}{\underbrace{(x+1)^2}_{\geq 0}(x+1)} > 0 \Rightarrow$$

a) $3x-1>0$
$\underline{x+1>0}$
$x>\frac{1}{3}$
$x>-1$
$\underline{x\in\langle\frac{1}{3},\infty\rangle}$

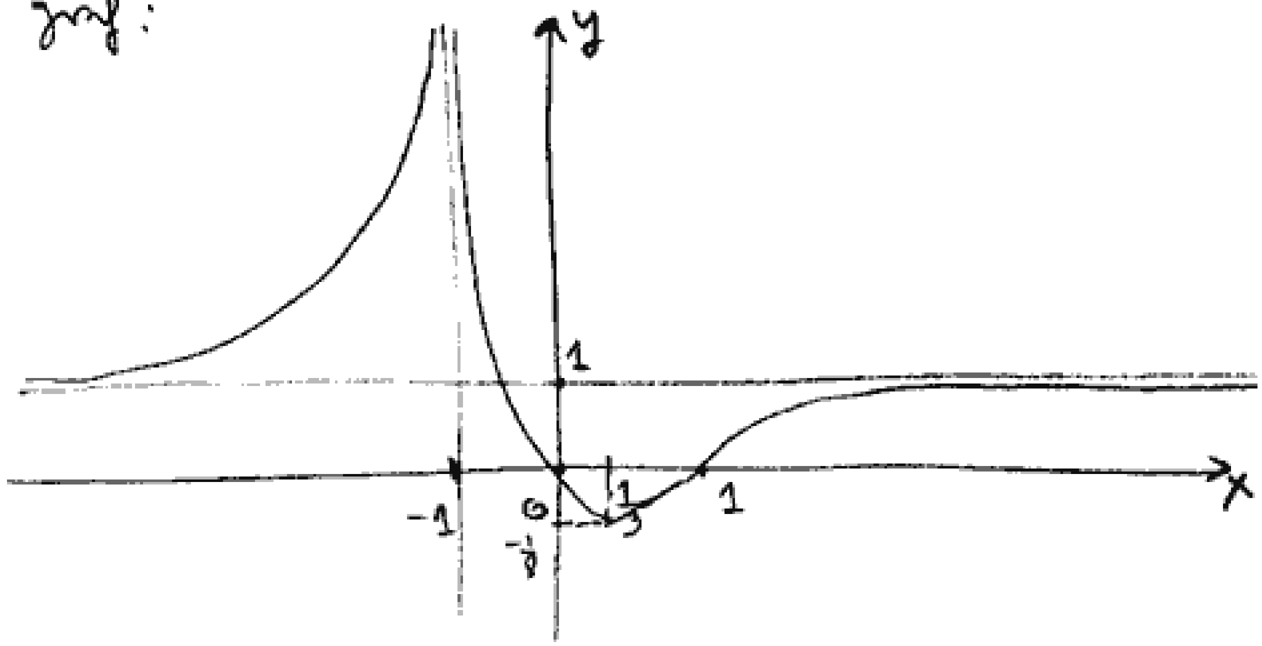
b) $3x-1<0$
$\underline{x+1<0}$
$x<\frac{1}{3}$
$x<-1$
$\underline{x\in\langle-\infty,-1\rangle}$

$\Rightarrow f$ raste na $\langle-\infty,-1\rangle \cup \langle\frac{1}{3},\infty\rangle$, a pada na $\langle-1,\frac{1}{3}\rangle$

6)

	$\langle-\infty,-1\rangle$	-1	$\langle-1,\frac{1}{3}\rangle$	$1/3$	$\langle\frac{1}{3},\infty\rangle$
f'	+	×	−	0	+
f	↗	×	↘	MIN	↗

7) graf:



Zadatak. Odredite sjecište pravaca $p_1 \ldots \begin{cases} 2x-3y+z=1 \\ x \quad\; -z=-4 \end{cases}$

$$p_2 \ldots \begin{cases} x-5y+3z=8 \\ 2x+y-z=-5 \end{cases}$$

i napišite jednadžbu pravca koji je okomit na p_1 i p_2, a prolazi sjecištem tih pravaca.

Rješenje. Odaberimo tri od ove četiri jednadžbe da nađemo presječnu točku – jedino rješenje sustava je $(-1,0,3)$, i to je tražena točka (uvrštavanje u četvrtu jednadžbu pokazuje da se p_1 i p_2 sijeku).
Kroz točku $(-1,0,3)$ treba povući pravac okomit na p_1 i p_2, što znači da za vektor smjera $\vec{s}$ tog pravca vrijedi $\vec{s}=\vec{s}_1\times\vec{s}_2$, gdje su $\vec{s}_1$ i $\vec{s}_2$ vektori smjera pravaca p_1 i p_2.

$$\vec{s}_1=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix}=3\vec{i}+3\vec{j}+3\vec{k}, \quad \vec{s}_2=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & 3 \\ 2 & 1 & -1 \end{vmatrix}=2\vec{i}+7\vec{j}+11\vec{k}$$

$$\vec{s}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 3 \\ 2 & 7 & 11 \end{vmatrix}=12\vec{i}-27\vec{j}+6\vec{k}$$

$\Rightarrow$ jednadžba traženog pravca je $\underline{\dfrac{x+1}{12}=\dfrac{y}{-27}=\dfrac{z-3}{15}}$.

Zadatak. Odredite domenu funkcije $f(x)=\log_x(4^x-6\cdot 2^x+8)$.

Rješenje. $x>0,\ x\neq 1$ (zbog baze log funkcije)

$4^x-6\cdot2^x+8>0$ (zbog argumenta log funkcije)

$(2^x)^2-6\cdot 2^x+8>0,\ 2^x=t$

$t^2-6t+8>0 \Rightarrow t_1<2$ ili $t_2>4$

$2^x<2$ ili $2^x>4 \Rightarrow x<1$ ili $x>2$

$\Rightarrow \boxed{\mathcal{D}(f)=\langle 0,1\rangle\cup\langle 2,\infty\rangle}$.

Zadatak. Bez upotrebe L'Hospitalovog pravila izračunajte $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{1-\sqrt{1-x}}$.

Rješenje.

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{1-\sqrt{1-x}}\cdot\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\cdot\frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}=$$

$$=\lim_{x\to 0}\frac{1+\sin x-(1-\sin x)}{1-(1-x)}\cdot\frac{1+\sqrt{1-x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}=$$

$$=\lim_{x\to 0}\underbrace{\frac{2\sin x}{x}}_{1}\cdot\frac{1+\sqrt{1-x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}=2\cdot 1\cdot\frac{1+\sqrt{1}}{\sqrt{1}+\sqrt{1}}=\boxed{2}$$

Zadatak. Dokažite da sve točke krivulje $y=\sin x$ u kojima je tangenta paralelna s pravcem $y=x$ imaju y-koordinatu nula.

Rješenje. Koeficijent smjera pravca $y=x$ je $k=1$. Da bi u točki (x_0, y_0) krivulje $y=\sin x$ tangenta bila paralelna s $y=x$, mora koeficijent smjera tangente biti $k_t=1$. No, koeficijent smjera je
$y'(x_0)=\cos x_0=1 \Rightarrow x_0=2k\pi, k\in\mathbb{Z}$. Sada je
$y_0=\sin x_0=\sin(2k\pi)=0$, što je i trebalo dokazati.

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{x}{\ln x}$.

Rješenje.

1) $\mathcal{D}(f)=\langle 0,\infty\rangle\setminus\{1\}$ (zbog $\ln x\neq 0$, $x>0$)

2) $\mathcal{N}(f)=\emptyset$

3) asimptote:

horizontalna $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{\ln x}\overset{L'H}{=}\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\lim_{x\to\infty} x=\infty$ nema

vertikalna $\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x}{\ln x}=\infty \Rightarrow \underline{x=1}$ vertikalna

kosa $y=kx+l$

$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{1}{\ln x}=0$ nema

4) $f'(x)=\frac{\ln x - x\cdot\frac{1}{x}}{(\ln x)^2}=\frac{\ln x-1}{\ln^2 x}=0 \quad \ln x=1\Rightarrow x=e$

$f''(x)=\frac{\frac{1}{x}\cdot\ln^2 x-(\ln x-1)\cdot 2\ln x\cdot\frac{1}{x}}{\ln^4 x}=\frac{1}{x}\cdot\frac{\cancel{\ln x}(\ln x-2\ln x+2)}{\ln^{\cancel{4}} x}=$

$=\frac{2-\ln x}{x\ln^3 x}$

$f''(e)=\frac{2-1}{e\cdot 1}=\frac{1}{e}>0\Rightarrow$ min.

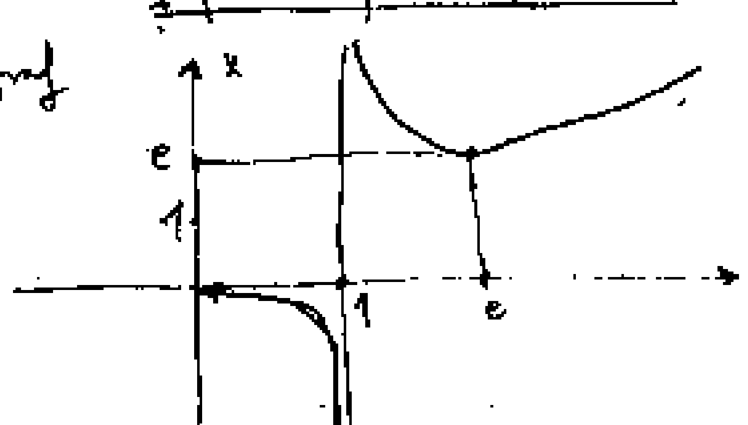
$f(e)=\frac{e}{\ln e}=e\rightarrow (e,e)$ je točka lok. minimuma

5) $f'(x)>0 \Rightarrow \ln x-1>0\Rightarrow \ln x>1\Rightarrow x>e$ rast $\Rightarrow x\in\langle 0,e\rangle\setminus\{1\}$ pad

6) tok:

	$\langle 0,1\rangle$	1	$\langle 1,e\rangle$	e	$\langle e,\infty\rangle$
f'	$-$	x	$-$	0	$+$
f	$\searrow$	x	$\searrow$	min	$\nearrow$

7) graf



! ponašanje u nuli:
$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{x}{\ln x}\overset{L'H}{=}\lim_{x\to 0+} x=0$

MATEMATIKA 1 - 6. SRPNJA 2004. - RJEŠENJA

1. Odredite domenu funkcije $f(x) = \arcsin\left(\log_2\left(\frac{x-1}{x+1}\right)\right)$.

Rješenje. uvjeti su: • $-1 \le \log_2 \frac{x-1}{x+1} \le 1$ (zbog arcsin)

$$\Rightarrow \frac{1}{2} \le \frac{x-1}{x+1} \le 2$$

• $\frac{x-1}{x+1} > 0$ (zbog $\log_2$)

Donji uvjet ne treba niti rješavati, jer je gornji „jači".

Rješavamo gornji uvjet:

$$\frac{1}{2} \le \frac{x-1}{x+1} \le 2$$

1. slučaj $x+1>0$, tj. $x>-1 \Rightarrow x \in \langle -1, \infty\rangle$

$\Rightarrow x+1 \le 2x-2$ i $x-1 \le 2x+2$

$3 \le x$ i $-3 \le x$

$\Rightarrow 3 \le x$, a rješavamo u skupu $\langle -1,\infty\rangle \Rightarrow$ $x \in [3,\infty\rangle$

2. slučaj $x+1<0$, tj. $x<-1 \Rightarrow x \in \langle -\infty,-1\rangle$

$\Rightarrow x+1 \ge 2x-2$ i $x-1 \ge 2x+2$

$3 \ge x$ i $-3 \ge x$

$\Rightarrow -3 \ge x$, a rješavamo u skupu $\langle -\infty,-1\rangle \Rightarrow$ $x \in \langle -\infty,-3]$

$$\Rightarrow \boxed{\mathcal{D}(f) = \langle -\infty,-3] \cup [3,\infty\rangle}$$

2. Bez upotrebe L'Hospitalovog pravila izračunajte $\lim\limits_{x\to\infty} x(\sqrt{x^2+1}-x)$.

Rješenje.
$$\lim_{x\to\infty} x(\sqrt{x^2+1}-x) = \lim_{x\to\infty}\left(x\cdot(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right) =$$

$$= \lim_{x\to\infty}\frac{x\cdot\left((\sqrt{x^2+1})^2-x^2\right)}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}+x} \overset{:x}{=} \lim_{x\to\infty}\frac{1}{\sqrt{1+\underbrace{\tfrac{1}{x^2}}_{0}}+1} = \boxed{\frac{1}{2}}$$

3. Izračunajte približno arctg 1.05.

Rješenje. $f(x) = \operatorname{arctg} x$, $x_0 = 1$, $\Delta x = 0.05 = \frac{5}{100} = \frac{1}{20}$

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x, \quad f'(x) = \frac{1}{1+x^2}$$

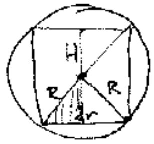
$$\Rightarrow \operatorname{arctg} 1.05 = f(1+0.05) \approx f(1) + f'(1)\cdot\frac{1}{20} = \operatorname{arctg} 1 + \frac{1}{2}\cdot\frac{1}{20} =$$

$$= \frac{\pi}{4} + \frac{1}{40} = \frac{10\pi+1}{40}$$

$$\Rightarrow \operatorname{arctg} 1.05 \approx \frac{10\pi+1}{40} \approx 0.803$$

4. Kugli polumjera R upišite valjak maksimalnog volumena.

Rješenje.



$\Rightarrow \quad \frac{H^2}{4} + r^2 = R^2$

$r^2 = R^2 - \frac{H^2}{4}$

$$V = r^2 \pi \cdot H = \left(R^2 - \frac{H^2}{4}\right) \cdot \pi H$$

$$V' = -\frac{\cancel{2}H}{\cancel{4}_2} \cdot \pi H + \left(R^2 - \frac{H^2}{4}\right) \cdot \pi = 0 \;\Big/ \cdot \frac{4}{\pi}$$

$$-2H^2 + 4R^2 - H^2 = 0$$

$$4R^2 = 3H^2 \Rightarrow \boxed{H = \frac{2\sqrt{3}}{3} R}$$

$$r^2 = R^2 - \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{3} R^2 = \frac{2}{3} R^2 \Rightarrow \boxed{r = \frac{\sqrt{6}}{3} R}$$

$$V_{max} = \frac{2}{3} R^2 \cdot \pi \cdot \frac{2\sqrt{3}}{3} R \Rightarrow \boxed{V_{max} = \frac{4\sqrt{3}}{9} R^3 \pi}$$

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5. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{x-3}{x^2-6x+5}$

Rješenje.

1) $D(f) = ?$ $\quad x^2 - 6x + 5 = 0 \Rightarrow x_1 = 1, x_2 = 5 \Rightarrow$ $\underline{D(f) = \mathbb{R} \setminus \langle 1, 5 \rangle}$

2) $N(f) = \{3\}$ $\quad x - 3 = 0 \Rightarrow x = 3$

3) asimptote: vertikalna: $x = 1, x = 5$

horizontalna: $\lim\limits_{x \to \infty} \frac{x-3}{x^2-6x+5} = 0 \Rightarrow \underline{y = 0}$

kosa: nema

4) $$f'(x) = \frac{1 \cdot (x^2 - 6x + 5) - (x-3) \cdot (2x-6)}{(x^2-6x+5)^2} = \frac{x^2 - \cancel{6x} + 5 - 2x^2 + 6x + \cancel{6x} - 18}{(x^2-6x+5)^2}$$

$$= \frac{-x^2 + 6x - 13}{(x^2-6x+5)^2} = 0 \Rightarrow \quad x^2 - 6x + 13 = 0$$

$D = 36 - 52 < 0 \Rightarrow$ nema kandidata za ekstrem

5) $$f''(x) = \frac{(-2x+6)(x^2-6x+5)^{\cancel{2}} - (-x^2+6x-13) \cdot 2\cancel{(x^2-6x+5)}(2x-6)}{(x^2-6x+5)^4}$$ kubna jednadžba

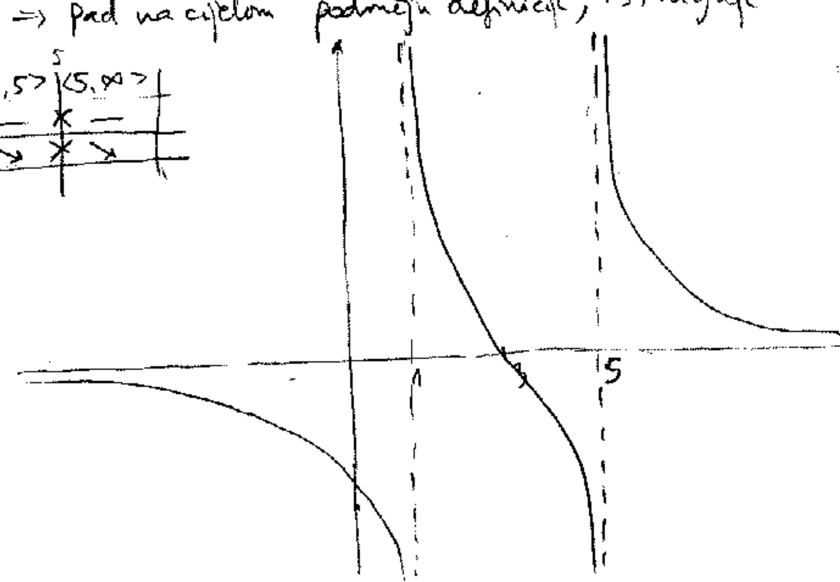
6) rast: $f'(x) > 0 \Rightarrow -x^2 + 6x - 13 > 0$.

$\Rightarrow$ pad na cijelom području definicije, rast nigdje

7)

	$\langle -\infty, 1 \rangle$	1	$\langle 1, 5 \rangle$	5	$\langle 5, \infty \rangle$
f'	−	×	−	×	−
f	↘	×	↘	×	↘

8) graf:



MATEMATIKA 1 – 13.7.4. – RJEŠENJA

Zadatak. Odredite točku simetričnu točki $(1,2,3)$ s obzirom na pravac

$$p \ldots \begin{cases} x+y-z=4 \\ x-y+z=2 \end{cases}$$

Rješenje. Nađimo kanonski oblik jednadžbe pravca:

$$\left.\begin{aligned} x+y-z&=4 \\ x-y+z&=2 \end{aligned}\right| + \Rightarrow x=3,\ z=t,\ y=4+t-3$$
$$y=1+t$$

$$\Rightarrow \frac{x-3}{0}=\frac{y-1}{1}=\frac{z-0}{1}$$

$$\Rightarrow \vec{s}=\vec{i}+\vec{j}$$

Položimo ravninu Π koja prolazi točkom $(1,2,3)$ i okomita je na $p \Rightarrow \vec{n}=\vec{s}=\vec{i}+\vec{j}$, pa je Π dano s $0\cdot(x-1)+1\cdot(y-2)+1\cdot(z-3)$

$\Rightarrow$ $\underline{y+z=5}$. Tražimo presjek $\Pi \cap p$:

$1+t+t=5 \Rightarrow t=2 \Rightarrow A'=(3,3,2)$. Znamo: $A''=(x,y,z)$ i

$$\boxed{A''=(5,4,1)} \Leftarrow \begin{cases} \frac{x+1}{2}=3 \\ \frac{y+2}{2}=3 \\ \frac{z+3}{2}=2 \end{cases}$$

Zadatak. Dokažite da je domena funkcije $f(x)=\arcsin\frac{x}{\sqrt{x^2+1}}$ čitav $\mathbb{R}$.

Rješenje. Jedini uvjet je $-1 \le \frac{x}{\sqrt{x^2+1}} \le 1$, tj. $\frac{|x|}{\sqrt{x^2+1}} \le 1 \Leftrightarrow |x| \le \sqrt{x^2+1}$

$\Leftrightarrow x^2 \le x^2+1$, što očito vrijedi $\Rightarrow D(f)=\mathbb{R}$.

Zadatak. Bez upotrebe L'Hospitalovog pravila izračunajte $\lim\limits_{x\to 1}\frac{\operatorname{tg}(\pi x)}{x-1}$.

Rješenje. Supst. $x-1=t$

$$\Rightarrow \lim_{t\to 0}\frac{\operatorname{tg}(\pi(t+1))}{t}=\lim_{t\to 0}\frac{\operatorname{tg}(\pi t+\pi)}{t}=\lim_{t\to 0}\frac{\operatorname{tg}\pi t+\overbrace{\operatorname{tg}\pi}^{0}}{t(1-\operatorname{tg}\pi t\,\operatorname{tg}\pi)}=$$

$$=\lim_{t\to 0}\frac{\operatorname{tg}\pi t}{t}=\lim_{t\to 0}\frac{\sin\pi t}{t}\cdot\frac{1}{\cos\pi t}=$$

$$=\lim_{t\to 0}\underbrace{\frac{\sin\pi t}{\pi t}}_{1}\cdot\pi\cdot\underbrace{\frac{1}{\cos\pi t}}_{1}=\boxed{\pi}.$$

Zadatak. Odredite tangente i normale na krivulju u točkama presjeka krivulje

$y=x^3-x^2-4x+4$ i x-osi.

Rješenje. $y=0 \Rightarrow \quad x^3-x^2-4x+4=0$

$$x^2(x-1)-4(x-1)=0$$

$$(x^2-4)(x-1)=0 \Rightarrow \quad x_1=-2,\ x_2=1,\ x_3=2$$

$$y_1=0,\ y_2=0,\ y_3=0$$

1. točka $(-2,0)$ koristimo formulu $y-y_1=y'(x_1)(x-x_1)$ tangenta

$$y-y_1=\frac{-1}{y'(x_1)}(x-x_1) \text{ normala}$$

$$y'=3x^2-2x-4 \Rightarrow y'(-2)=12 \Rightarrow \quad y=12x+24 \text{ tangenta}, \quad y=-\frac{1}{12}x-\frac{1}{6} \text{ normala}$$

2. točka $(1,0)$ $y'(1)=-3 \Rightarrow \quad y=-3x+3$ tangenta, $y=\frac{1}{3}x-\frac{1}{3}$ normala

3. točka $(2,0)$ $y'(2)=4 \Rightarrow \quad y=4x-8$ tangenta, $y=-\frac{1}{4}x+\frac{1}{2}$ normala

---

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{x^2-4x+5}{x-2}$.

Rješenje.
- $D(f)=\mathbb{R}\setminus\{2\}$
- $N(f)=?$ $\quad x^2-4x+5=0 \Rightarrow$ nema realnih rješenja $\Rightarrow N(f)=\emptyset$
- asimptote: vertikalna $x=2$

  horizontalna $\lim\limits_{x\to\infty} f(x)=\infty \to$ nema

  kosa $y=kx+l$

  $$k=\lim_{x\to\infty}\frac{x^2-4x+5}{x(x-2)}=1$$

  $$l=\lim_{x\to\infty}\left(\frac{x^2-4x+5}{x-2}-x\right)=\lim_{x\to\infty}\frac{-2x+5}{x-2}=-2$$

  $\Rightarrow y=x-2$ kosa
- $f'(x)=\frac{(2x-4)(x-2)-(x^2-4x+5)}{(x-2)^2}=\frac{x^2-4x+3}{(x-2)^2}=0 \Rightarrow x^2-4x+3=0 \Rightarrow x_1=1,\ x_2=3$ kandidati ekstrem
- $f''(x)=\frac{(2x-4)(x-2)^{\cancel{2}}-(x^2-4x+3)\cdot 2\cancel{(x-2)}}{(x-2)^{\cancel{4}}}=\frac{2}{(x-2)^3}$

$f''(1)=-2<0 \Rightarrow (1,f(1))=(1,-2)$ max

$f''(3)=2>0 \Rightarrow (3,f(3))=(3,2)$ min

- rast: $f'(x)>0 \Rightarrow \frac{x^2-4x+3}{\underbrace{(x-2)^2}_{+}}>0 \Rightarrow x^2-4x+3>0 \Rightarrow x\in\langle-\infty,1\rangle\cup\langle3,\infty\rangle$ rast

  $\Rightarrow x\in\langle1,3\rangle$ pad

- tok:

	$\langle-\infty,1\rangle$	1	$\langle1,2\rangle$	2	$\langle2,3\rangle$	3	$\langle3,\infty\rangle$
f'	+	0	−	×	−	0	+
f	↗	MAX	↘	×	↘	MIN	↗

- graf:

MATEMATIKA 1 / 9.9.4. / RJEŠENJA

1. Nađite jednadžbu ravnine koja sadrži pravac $p_1 \ldots \begin{cases} 3x+y-z+5=0 \\ -x+y+z-3=0 \end{cases}$,

a paralelna je s pravcem $p_2 \ldots \begin{cases} x+y-z+5=0 \\ x+3y=0. \end{cases}$

Rješenje. Tražimo Π s vektorom normale $\vec{n}$ i točkom T.

$\vec{n} = ?$

$\vec{n} = \vec{s_1} \times \vec{s_2}$, gdje je $\vec{s_1}$ vektor smjera pravca p_1, a $\vec{s_2}$ vektor smjera pravca p_2.

$$\vec{s_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\vec{s_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 3 & 0 \end{vmatrix} = 3\vec{i} - \vec{j} + 2\vec{k}$$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \underline{\underline{8\vec{j} + 4\vec{k}}}$$

$T = ?$ Možemo uzeti proizvoljnu točku s pravca p_1. Neka je $z=0$, uvrstimo u $p_1 \Rightarrow \begin{rcases} 3x+y=-5 \\ -x+y=3 \end{rcases} \Rightarrow x=-2, y=1 \Rightarrow \underline{\underline{T=(-2,1,0)}}$

$\Pi \ldots \; 8(y-1)+4z=0 \; / : 4 \Rightarrow \boxed{2y+z=2}$

---

2. Odredite domenu funkcije $f(x) = \sqrt{(3\log^2_{\frac{1}{8}} x - 4\log_{\frac{1}{8}} x + 1)(4x^2-1)}$

Rješenje. • $\boxed{x>0}$ (zbog log)

• $(3\log^2_{\frac{1}{8}} x - 4\log_{\frac{1}{8}} x + 1)(4x^2-1) \geq 0$ (zbog $\sqrt{\;\;}$)

Imamo dvije mogućnosti:

I) $3\log^2_{\frac{1}{8}} x - 4\log_{\frac{1}{8}} x + 1 \geq 0$, $\; 4x^2-1 \geq 0$

II) $3\log^2_{\frac{1}{8}} x - 4\log_{\frac{1}{8}} x + 1 \leq 0$, $\; 4x^2-1 \leq 0$

I) $4x^2-1 \geq 0 \Rightarrow x^2 \geq \frac{1}{4} \Rightarrow |x| \geq \frac{1}{2}$; zbog $x>0$ je $\boxed{x \geq \frac{1}{2}}$

$3\log^2_{\frac{1}{8}} x - 4\log_{\frac{1}{8}} x + 1 \geq 0$ (uvedimo supst. $\log_{\frac{1}{8}} x = t$)

$\Rightarrow 3t^2-4t+1 \geq 0 \Rightarrow t \leq \frac{1}{3}$ ili $t \geq 1$

$\log_{\frac{1}{8}} x \leq \frac{1}{3} \; / \left(\frac{1}{8}\right)^{\square} \Rightarrow x \geq \left(\frac{1}{8}\right)^{1/3}$, $\; x \geq \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$

$\log_{\frac{1}{8}} x \geq 1 \; / \left(\frac{1}{8}\right)^{\square} \Rightarrow x \leq \frac{1}{8}$

$\Rightarrow \boxed{x \in \langle -\infty, \frac{1}{8}] \cup [\frac{1}{2}, \infty\rangle} \Rightarrow \boxed{x \geq \frac{1}{2}}$

II) $4x^2-1 \leq 0 \Rightarrow x^2 \leq \frac{1}{4} \Rightarrow |x| \leq \frac{1}{2}$;

zbog $x>0$ je $\boxed{x \in \langle 0, \frac{1}{2}]}$

$3t^2-4t+1 \leq 0 \Rightarrow \frac{1}{3} \leq t \leq 1 \Rightarrow$

$\frac{1}{3} \leq \log_{\frac{1}{8}} x \leq 1 \; / \left(\frac{1}{8}\right)^{\square} \Rightarrow$

$\boxed{\frac{1}{2} \geq x \geq \frac{1}{8}} \Rightarrow \boxed{x \in [\frac{1}{8}, \frac{1}{2}]}$

Ukupno rješenje pod I) + II) je $x \in [\frac{1}{3}, \infty\rangle$, pa je

$$\boxed{\mathcal{D}(f) = [\tfrac{1}{3}, \infty\rangle}$$

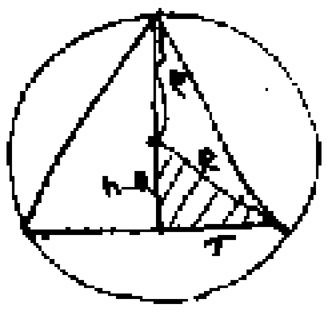
3. Izračunajte bez L'Hospitalovog pravila $\lim\limits_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3}$.

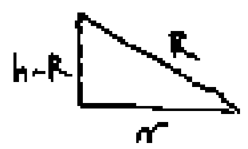
Rješenje. $\lim\limits_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \frac{1-\cos x}{\cos x} \cdot \frac{1}{x^2} =$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{1} \cdot \underbrace{\frac{+2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2 \cdot 4}}_{1} \cdot \underbrace{\frac{1}{\cos x}}_{1} = \boxed{+\frac{1}{2}}$$

4. U sferu zadanog polumjera R upišite stožac maksimalnog volumena.

Rješenje.





$\Rightarrow R^2 = (h-R)^2 + r^2$

$\cancel{R^2} = h^2 - 2hR + \cancel{R^2} + r^2 \Rightarrow \underline{r^2 = 2hR - h^2}$

$V = \frac{1}{3} r^2 \pi \cdot h = \frac{1}{3}\pi \cdot (2hR - h^2)h =$

$= \frac{1}{3}\pi \underbrace{(-h^3 + 2h^2R)}_{f(h)}$

$f'(h) = -3h^2 + 4Rh = 0 \Rightarrow 3h = 4R \Rightarrow \boxed{h = \frac{4}{3}R}$

$r^2 = 2 \cdot \frac{4}{3}R \cdot R - \frac{16}{9}R^2 = \frac{8}{9}R^2 \Rightarrow \boxed{r = \frac{2\sqrt{2}}{3}R}$

$V_{max} = \frac{1}{3} \cdot \frac{8}{9}R^2 \pi \cdot \frac{4}{3}R \Rightarrow \boxed{V_{max} = \frac{32}{81}R^3\pi}$

5. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{2x-3}{(x-2)^2}$.

Rješenje. 1) $\mathcal{D}(f) = \mathbb{R} \setminus \{2\}$

2) $\mathcal{N}(f) = \{3/2\}$

3) asimptote

V.A. $\boxed{x=2}$

H.A. $\lim\limits_{x\to\infty} f(x) = 0 \Rightarrow \boxed{y=0}$

K.A. nema

4) $f'(x) = \frac{2(x-2)^{\cancel{2}} - (2x-3)\cdot 2\cancel{(x-2)}}{(x-2)^{\cancel{4}3}} = \frac{2x-4-4x+6}{(x-2)^3} = \frac{2-2x}{(x-2)^3} = 0 \Rightarrow \boxed{x=1}$

5) $f''(x) = \frac{-2(x-2)^{\cancel{3}} - (2-2x)\cdot 3\cancel{(x-2)^2}}{(x-2)^{\cancel{6}4}} = \frac{-2x+4-6+6x}{(x-2)^4} = \frac{4x-2}{(x-2)^4} \Rightarrow f''(1) = 2 > 0 \Rightarrow \boxed{\text{MIN}}$

$f(1) = -1 \Rightarrow \boxed{(1,-1)}$

6) $\underset{\text{rast}}{f'(x) > 0} \rightarrow \frac{2-2x}{\underbrace{(x-2)^2}_{\text{uvijek}}(x-2)} > 0 \Rightarrow$ a) $\left.\begin{array}{l} 2-2x>0 \Rightarrow x<1 \\ x-2>0 \Rightarrow x>2 \end{array}\right\}$ nema rj.

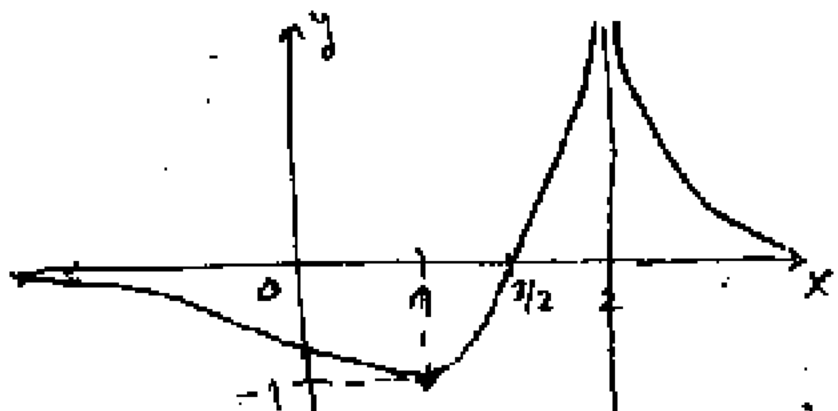
b) $\left.\begin{array}{l} 2-2x<0 \Rightarrow x>1 \\ x-2<0 \Rightarrow x<2 \end{array}\right\} \Rightarrow \boxed{x \in \langle 1,2\rangle}$

$\Rightarrow$ pad: $\boxed{x \in \langle -\infty, 1\rangle \cup \langle 2, \infty\rangle}$

7) tok

	$\langle -\infty,1\rangle$	1	$\langle 1,2\rangle$	2	$\langle 2,\infty\rangle$
f'	−	●	+	×	−
f	↘	MIN	↗	×	↘

8) graf



MATEMATIKA 1 – 23. RUJNA 2004. – RJEŠENJA

1. Odredite $x \in \mathbb{R}$ tako da vektori
$$\vec{a} = (2x-6)\vec{i} + 4\vec{j} - 3\vec{k}$$
$$\vec{b} = (3x-1)\vec{i} + 2\vec{j} + 2\vec{k}$$
$$\vec{c} = (3-8x)\vec{i} + (x-2)\vec{j} - 3x\vec{k}$$
budu komplanarni, te u tom slučaju izrazite vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rješenje. $\vec{a}, \vec{b}$ i $\vec{c}$ su komplanarni ako je $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$, tj. ako je

$$\begin{vmatrix} 2x-6 & 4 & -3 \\ 3x-1 & 2 & 2 \\ 3-8x & x-2 & -3x \end{vmatrix} = (2x-6)(-6x-2x+4) - 4(-9x+3x-6+16x) - 3((3x-1)(x-2)-6+16x) = \dots = 11x^2-47x+12 = 0$$

$\Rightarrow \quad x_1 = \frac{3}{11},\ x_2 = 4 \quad \to$ 2 rješenja

1. rješenje: $x_1 = \frac{3}{11} \Rightarrow$
$$\vec{a} = -\frac{60}{11}\vec{i} + 4\vec{j} - 3\vec{k}$$
$$\vec{b} = -\frac{2}{11}\vec{i} + 2\vec{j} + 2\vec{k}$$
$$\vec{c} = \frac{9}{11}\vec{i} - \frac{19}{11}\vec{j} - \frac{9}{11}\vec{k}$$

$$\vec{c} = \alpha\vec{a} + \beta\vec{b} \Rightarrow \left.\begin{array}{l} \frac{9}{11} = -\frac{60}{11}\alpha - \frac{2}{11}\beta \\ -\frac{19}{11} = 4\alpha + 2\beta \\ -\frac{9}{11} = -3\alpha + 2\beta \end{array}\right\} \Rightarrow \alpha = \frac{-10}{77},\ \beta = -\frac{93}{154}$$
$$\Rightarrow \boxed{\vec{c} = \frac{-10}{77}\vec{a} - \frac{93}{154}\vec{b}}$$

2. rješenje: $x_2 = 4 \Rightarrow$
$$\vec{a} = 2\vec{i} + 4\vec{j} - 3\vec{k}$$
$$\vec{b} = 11\vec{i} + 2\vec{j} + 2\vec{k}$$
$$\vec{c} = -29\vec{i} + 2\vec{j} - 12\vec{k}$$

$$\vec{c} = \alpha\vec{a} + \beta\vec{b} \Rightarrow \left.\begin{array}{l} -29 = 2\alpha + 11\beta \\ 2 = 4\alpha + 2\beta \\ -12 = -3\alpha + 2\beta \end{array}\right\} \Rightarrow \alpha = 2,\ \beta = -3$$
$$\Rightarrow \boxed{\vec{c} = 2\vec{a} - 3\vec{b}}$$

---

2. Odredite domenu funkcije $f(x) = \sqrt{\pi^2 - (3\arcsin x)^2}$.

Rješenje.
- $\boxed{-1 \le x \le 1}$ (zbog arcsin)
- $\pi^2 - (3\arcsin x)^2 \ge 0$ (zbog $\sqrt{\ }$)

$\Rightarrow (3\arcsin x)^2 \le \pi^2$

$\Rightarrow |3\arcsin x| \le \pi \to -\pi \le 3\arcsin x \le \pi \ |:3$

$-\frac{\pi}{3} \le \arcsin x \le \frac{\pi}{3} \Rightarrow \boxed{-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}}$

$\Rightarrow \mathcal{D}(f) = [-1, 1] \cap [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}] \to \boxed{\mathcal{D}(f) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]}$

---

3. Bez korištenja L'Hospitalovog pravila izračunajte $\lim\limits_{x\to 1} \frac{1-x^2}{\sin \pi x}$.

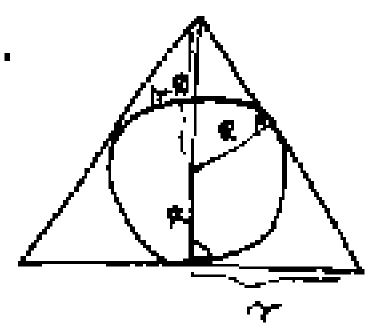
Rješenje. Uvodimo supstituciju $x - 1 = t$

$$\Rightarrow \lim_{t\to 0} \frac{1-(t+1)^2}{\sin(\pi(t+1))} = \lim_{t\to 0} \frac{\cancel{1}-t^2-2t-\cancel{1}}{\sin(\pi t+\pi)} = \lim_{t\to 0} \frac{-t^2-2t}{-\sin \pi t} = \lim_{t\to 0} \frac{t(t+2)}{\sin \pi t} =$$

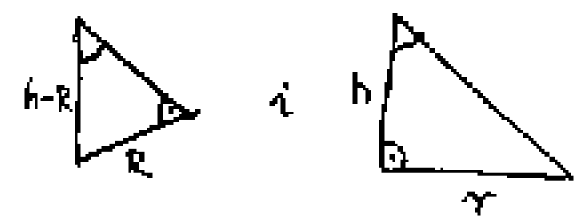
$$= \lim_{t\to 0} \frac{\pi t}{\sin \pi t} \cdot \frac{1}{\pi} \cdot (t+2) = \boxed{\frac{2}{\pi}}.$$

Zadatak. Kugli polumjera R opišite stožac minimalnog volumena.

Rješenje.



Imamo slične trokute:



$$\Rightarrow \frac{R}{r} = \frac{h-R}{\sqrt{h^2+r^2}}$$

$$\Rightarrow R\sqrt{h^2+r^2} = r(h-R) \Rightarrow R^2(h^2+r^2) = r^2(h^2+R^2-2hR)$$

$$R^2h^2+\cancel{R^2r^2} = r^2h^2+\cancel{r^2R^2}-2hRr^2$$

$$R^2h^2 = r^2(h^2-2hR)$$

$$\Rightarrow r^2 = \frac{R^2h^2}{h^2-2hR} = \frac{R^2h}{h-2R}$$

$$V_S = \frac{1}{3} r^2 \pi \cdot h$$

$$V_S = \frac{1}{3}\pi \cdot h \cdot \frac{R^2h}{h-2R} = \frac{\pi R^2}{3} \cdot \underbrace{\frac{h^2}{h-2R}}_{f(h)}$$

$$f'(h) = \frac{2h(h-2R)-h^2\cdot 1}{(h-2R)^2} = 0$$

$$h(2h-4R-h)=0 \quad h\neq 0$$

$$\Rightarrow \boxed{h=4R}$$

$$\Rightarrow r^2 = \frac{R^2\cdot 4R}{4R-2R} = \frac{4R^3}{2R} = 2R^2$$

$$\Rightarrow \boxed{r=R\sqrt{2}}$$

$$\boxed{(V_S)_{min} = \frac{1}{3}\pi R^2 \cdot \frac{(4R)^2}{4R-2R} = \frac{8}{3}\pi R^3}$$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{x^2-3x}{x-1}$.

Rješenje. $D(f) = \mathbb{R}\setminus\{1\}$

$N(f) = \{0,3\}$

asimptote: vertikalna $x=1$

horizontalna nema

kosa: $k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x-3}{x-1} = 1 \Rightarrow l = \lim_{x\to\infty}(f(x)-x) = \lim_{x\to\infty} \frac{x^2-3x-\cancel{x^2}+x}{x-1} =$

$= \lim_{x\to\infty} \frac{-2x}{x-1} = -2 \Rightarrow y = x-2$

ekstremi: $f'(x) = \frac{(2x-3)(x-1)-(x^2-3x)\cdot 1}{(x-1)^2} = \frac{x^2-2x+3}{(x-1)^2} = 0 \Rightarrow$ nema kandidata za ekstreme

$(D = 4-4\cdot 3 < 0\ !)$

rast-pad: rast $f'(x) > 0$

zbog nazivnika $(x-1)^2$ mora biti

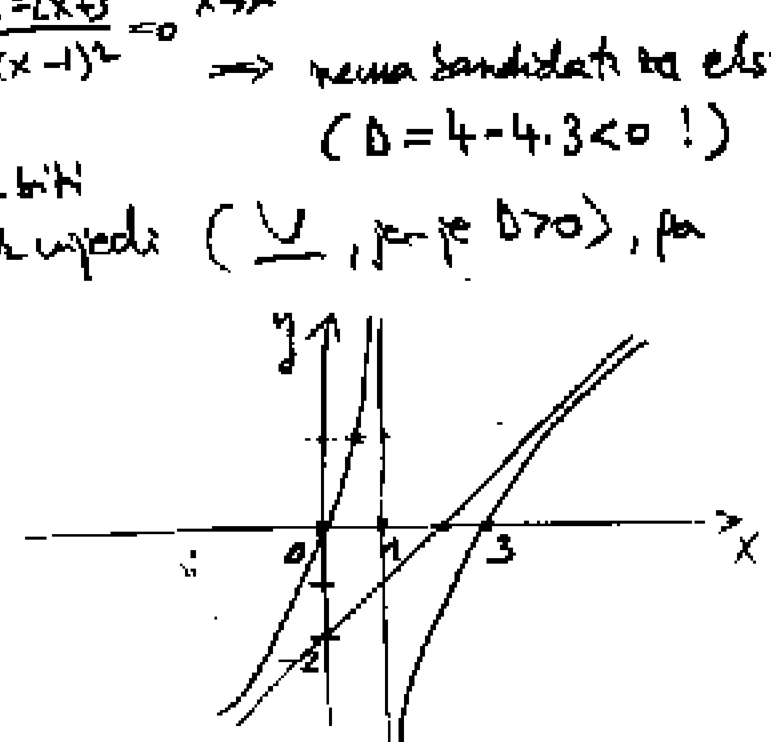
$x^2-2x+3>0$, no to uvijek vrijedi ($\cup$, jer je $D>0$), pa

f raste na cijeloj domeni

tok:

	$\langle -\infty, 1\rangle$		$\langle 1, \infty\rangle$
f'	+	$\times$	+
f	$\nearrow$	$\times$	$\nearrow$

graf:



MATEMATIKA 1 – 1. LISTOPADA 2004 – RJEŠENJE

1. Napišite jednadžbu ravnine koja sadrži pravce

$$p_1 \ldots \frac{x-3}{2} = \frac{y-2}{1} = \frac{z-1}{2}$$

$$p_2 \ldots \frac{x-3}{1} = \frac{y-2}{2} = \frac{z-1}{3}.$$

Rješenje. $p_1 \cap p_2 = \{(3,2,1)\}$, a $\vec{n} = \vec{s}_1 \times \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\vec{i} - 4\vec{j} + 3\vec{k}$, pa je

$$-(x-3) - 4(y-2) + 3(z-1) = 0 \quad / \cdot (-1)$$

$$x - 3 + 4y - 8 - 3z + 3 = 0$$

$$\boxed{x + 4y - 3z = 8}$$

2. Odredite domenu funkcije $f(x) = \arccos \frac{x^2-4}{x^2-1} + \arcsin \frac{5-x^2}{x^2-3}$.

Rješenje.

a) $-1 \le \frac{x^2-4}{x^2-1} \le 1$ (zbog arccos)

b) $-1 \le \frac{5-x^2}{x^2-3} \le 1$ (zbog arcsin)

a) 1) $x^2-1 > 0$ množenje nazivnikom ne mijenja predznak:

$$-x^2+1 \le x^2-4 \le x^2-1 \quad \Rightarrow \quad 5 \le 2x^2 \Rightarrow \boxed{|x| \ge \sqrt{\tfrac{5}{2}}}$$

ovo očito vrijedi

2) $x^2-1 < 0$ množenje nazivnikom mijenja predznak:

$$-x^2+1 \ge x^2-4 \ge x^2-1 \quad \Rightarrow \text{nema rješenja}$$

očito ne vrijedi

b) 1) $x^2-3 > 0 \Rightarrow -x^2+3 \le 5-x^2 \le x^2-3 \Rightarrow 8 \le 2x^2 \Rightarrow x^2 \ge 4 \Rightarrow \boxed{|x| \ge 2}$

očito vrijedi

2) $x^2-3 < 0 \Rightarrow -x^2+3 \ge 5-x^2 \ge x^2-3 \Rightarrow$ nema rješenja

očito ne vrijedi

$$\Rightarrow \boxed{\mathcal{D}(f) = \langle -\infty, -2] \cup [2, \infty\rangle}$$

4. Bez korištenja L'Hospitalovog pravila izračunajte $\lim_{x\to 0} \left(\frac{x^2 \sin\frac{1}{x}}{\sin x} \right)$.

Rješenje. $\lim_{x\to 0} \left(\frac{x^2 \sin\frac{1}{x}}{\sin x} \right) = \lim_{x\to 0} \underbrace{\frac{x}{\sin x}}_{1} \cdot \underbrace{x}_{0} \cdot \sin\frac{1}{x} = \underline{\underline{0}}$ — po apsolutnoj vrijednosti manje od

3. Približno izračunajte $255.94^{0.25}$.

Rješenje. Koristimo formulu $f(x_0+\Delta x) \approx \Delta x f'(x_0) + f(x_0)$, gdje je

$$f(x) = \sqrt[4]{x}, \quad x_0 = 256, \quad \Delta x = -0.06 = -\frac{6}{100} = \frac{-3}{50}$$

$$\Rightarrow \quad 255.94^{0.25} \approx -\frac{3}{50}\cdot\frac{1}{4}(256)^{-3/4} + \sqrt[4]{256} = -\frac{3}{50}\cdot\frac{1}{4}\cdot\frac{1}{4^3} + 4 =$$

$$= 4 - \frac{3}{12800} = \frac{51197}{12800} \approx 3.999765\ldots$$

__Zadatak.__ Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{4x-4}{x^2-2x+5}$.

__Rješenje.__ $\mathcal{D}(f) = ?$ $\quad x^2-2x+5=0 \quad D = 4-20=-16<0 \Rightarrow \underline{\mathcal{D}(f) = \mathbb{R}}$

$\mathcal{N}(f) = \{1\}$

asimptote: v.a. nema

h.a. $\lim\limits_{x\to\infty} \frac{4x-4}{x^2-2x+5} = \lim\limits_{x\to\infty} \frac{\frac{4}{x}-\frac{4}{x^2}}{1-\frac{2}{x}+\frac{5}{x^2}} = \frac{0}{1} = 0 \Rightarrow \underline{y=0 \text{ V.A}}$

k.a. nema

$$f'(x) = \frac{4(x^2-2x+5)-(4x-4)\cdot(2x-2)}{(x^2-2x+5)^2} = 0 \Rightarrow \quad x^2-2x+5-(x-1)(2x-2)=0$$

$$x^2-\cancel{2x}+5-2x^2+\cancel{2x}+2x-2=0$$

$$-x^2+2x+3=0$$

$$x^2-2x-3=0$$

$$x_{1,2} = \frac{2\pm\sqrt{4+12}}{2} = \frac{2\pm 4}{2} = 1\pm$$

$$x_1=-1,\ x_2=3$$

$$f'(x) = \frac{-4x^2+8x+12}{(x^2-2x+5)^2}$$

$$f''(x) = \frac{(-8x+8)(x^2-2x+5)^{\cancel{2}\,1} - (-4x^2+8x+12)\cdot 2\cancel{(x^2-2x+5)}\cdot(2x-2)}{(x^2-2x+5)^{\cancel{4}\,3}} =$$

$$= \frac{8\left(-(x-1)(x^2-2x+5)-(x-1)(-2x^2+4x+6)\right)}{(x^2-2x+5)^3} =$$

$$= \frac{8\left(-x^3+\cancel{2x^2}-5x+x^2-2x+5+2x^3-4x^2-6x-\cancel{2x^2}+4x+6\right)}{(x^2-2x+5)^3} =$$

$$= \frac{8(x^3-3x^2-9x+11)}{(x^2-2x+5)^3} \Rightarrow \quad f''(-1)>0 \Rightarrow (-1, f(-1)) \text{ min}$$

$$f''(3)<0 \Rightarrow (3, f(3)) \text{ max}$$

$$f(-1) = -1 \Rightarrow (-1,-1) \text{ min}$$

$$f(3) = 1 \Rightarrow (3,1) \text{ max}$$

rast-pad: rast $f'(x)>0$

$$\frac{-4x^2+8x+12}{\underbrace{(x^2-2x+5)^2}_{>0}} > 0$$

$$\Rightarrow -4x^2+8x+12>0 \ /:(-4) \Rightarrow \quad x^2-8x-12<0$$

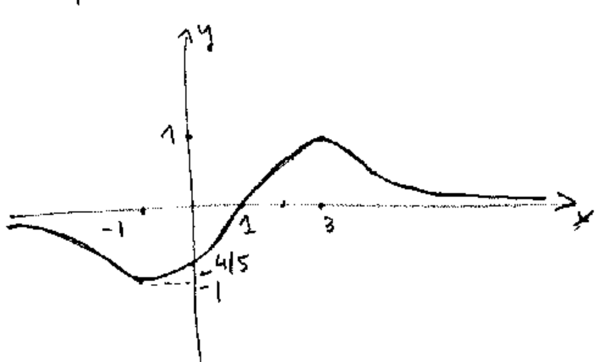
$\Rightarrow \boxed{x\in\langle -1,3\rangle}$ rast

$\Rightarrow$ pad: $\boxed{x\in\langle -\infty,-1\rangle \cup \langle 3,\infty\rangle}$

tok:

	$\langle -\infty,-1\rangle$	-1	$\langle -1,3\rangle$	3	$\langle 3,\infty\rangle$
f'	$-$	0	$+$	0	$-$
f	$\searrow$	MIN	$\nearrow$	MAX	$\searrow$

graf:



MATEMATIKA 1 - 11.12.04. - RJEŠENJA A

1. Gaussovom metodom eliminacije riješite sustav:

$$\begin{aligned} 2x_1+3x_2-x_3-x_4&=1\\ x_1+x_2-x_3-2x_4&=-8\\ x_1-x_2+2x_3+2x_4&=13\\ 2x_1+2x_2-x_3-2x_4&=-5. \end{aligned}$$

Rješenje.

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & -1 & 1\\ 1 & 1 & -1 & -2 & -8\\ 1 & -1 & 2 & 2 & 13\\ 2 & 2 & -1 & -2 & -5 \end{array}\right] \underset{I \leftrightarrow II}{\sim} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & -2 & -8\\ 2 & 3 & -1 & -1 & 1\\ 1 & -1 & 2 & 2 & 13\\ 2 & 2 & -1 & -2 & -5 \end{array}\right] \underset{\substack{I\cdot(-2)+II\to II\\ I\cdot(-1)+III\to III\\ I\cdot(-2)+IV\to IV}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & -8\\ 0 & \boxed{1} & 1 & 3 & 17\\ 0 & -2 & 3 & 4 & 21\\ 0 & 0 & 1 & 2 & 11 \end{array}\right] \underset{\substack{II\cdot(-1)+I\to I\\ II\cdot 2+III\to III}}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25\\ 0 & 1 & 1 & 3 & 17\\ 0 & 0 & 5 & 10 & 55\\ 0 & 0 & 1 & 2 & 11 \end{array}\right] \underset{III\cdot\frac{1}{5}\to III}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25\\ 0 & 1 & 1 & 3 & 17\\ 0 & 0 & 1 & 2 & 11\\ 0 & 0 & 1 & 2 & 11 \end{array}\right] \sim$$

3. i 4. jednadžba su jednake, pa jednu (četvrtu) možemo prekrižiti

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25\\ 0 & 1 & 1 & 3 & 17\\ 0 & 0 & \boxed{1} & 2 & 11 \end{array}\right] \underset{\substack{III\cdot 2+I\to I\\ III\cdot(-1)+II\to II}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3\\ 0 & 1 & 0 & 1 & 6\\ 0 & 0 & 1 & 2 & 11 \end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} x_1-x_4&=-3\\ x_2+x_4&=6\\ x_3+2x_4&=11 \end{aligned}\right\}$$

proglasimo x_4 parametrom, tj. $x_4=t,\ t\in\mathbb{R}$, pa imamo

$$\begin{aligned} x_1&=-3+t\\ x_2&=6-t\\ x_3&=11-2t \end{aligned}$$

$\Rightarrow$ sustav ima beskonačno mnogo rješenja: $\underline{(-3+t,\,6-t,\,11-2t,\,t),\ t\in\mathbb{R}}$.

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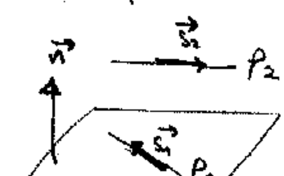
2. Nađite jednadžbu ravnine koja sadrži pravac $p_1 \ldots \begin{cases} 3x+y-z+5=0\\ -x+y+z-3=0 \end{cases}$, a paralelna je s pravcem $p_2 \ldots \begin{cases} x+y-z+5=0\\ x+3y=0. \end{cases}$

Rješenje.



Označimo s Π ravninu koju tražimo te neka je njen vektor normale $\vec{n}$. Očito $\vec{n}\perp\vec{s_1}$ i $\vec{n}\perp$ gdje su $\vec{s_1}$ i $\vec{s_2}$ vektori smjera pravca p_1 i p_2, redom, p možemo uzeti $\vec{n}=\vec{s_1}\times\vec{s_2}$.

Kako je p_1 zadan kao presjek dvije ravnine, to će biti $\vec{s_1}\perp\vec{n_1}$, $\vec{s_1}\perp\vec{n_2}$, gdje s $\vec{n_1}$ i $\vec{n_2}$ vektori normala tih ravnina, pa imamo $\vec{s_1}=\vec{n_1}\times\vec{n_2}$. Analogno je $\vec{s_2}=\vec{n_3}\times\vec{n_4}$, gdje su $\vec{n_3}$ i $\vec{n_4}$ vektori normala ravnina koje određuju p_2.

$\Rightarrow \quad \vec{s}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} + 4\vec{k}$, $\vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 3 & 0 \end{vmatrix} = 3\vec{i} - \vec{j} + 2\vec{k}$, pa

je $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = 2\vec{j} + 4\vec{k}$.

Još nam treba točka T_0 iz Π. Možemo uzeti proizvoljnu točku s pravca p_2 jer $p_2 \in \Pi$.
To će biti bilo koja točka (x_0, y_0, z_0) koja zadovoljava sustav $\begin{cases} 3x+y-z+5=0 \\ -x+y+z-3=0 \end{cases}$. Možemo

uzeti $\underline{z_0=0}$, pa imamo $\begin{matrix} 3x_0+y_0+5=0 \\ -x_0+y_0-3=0 \end{matrix} \Big/ - \Rightarrow 4x_0 = -8 \Rightarrow \underline{x_0=-2} \Rightarrow \underline{y_0=1}$

$\Rightarrow T_0 = (-2, 1, 0)$, pa jednadžba ravnine glasi: $0\cdot(x+2) + 8(y-1) + 4(z-0) = 0 \;/ :4$

$\boxed{2y+z=2}$

3. Odredite domenu funkcije $f(x) = \sqrt{\log_{|x|}(|x|-x)}$.

Rješenje: • $\underbrace{|x|>0}_{x\in\mathbb{R}\setminus\{0\}}$, $|x|\neq 1 \Rightarrow x \notin \{-1,1\}$ (zbog baze log funkcije)

$\Rightarrow \underline{x \in \mathbb{R}\setminus\{-1,0,1\}}$

• $|x|-x>0$ (zbog argumenta log funkcije)

$\Rightarrow |x|>x \rightarrow$ vrijedi za sve $\underline{x \in \langle -\infty, 0\rangle}$ (za $x\in[0,\infty\rangle$ je $|x|=x$!)

• $\log_{|x|}(|x|-x) \geq 0$ (zbog $\sqrt{\;}$ funkcije)

Kako je $x \in \langle -\infty, 0\rangle \setminus\{-1\}$, to gornja nejednakost glasi $\log_{-x}(-x-x) \geq 0$

$\log_{-x}(-2x) \geq 0$

a) ako je $-x \in \langle 0,1\rangle$, tj. $x \in \langle -1, 0\rangle$, onda imamo $-2x \leq (-x)^0$

$-2x \leq 1$

$x \geq -\frac{1}{2} \Rightarrow \boxed{x \in [-\frac{1}{2},}$

b) ako je $-x \in \langle 1, \infty\rangle$, tj. $x \in \langle -\infty, -1\rangle$, onda imamo $-2x \geq (-x)^0$

$-2x \geq 1$

$x \leq -\frac{1}{2} \Rightarrow \boxed{x \in \langle -\infty, -}$

$\rightarrow \boxed{\mathcal{D}(f) = \langle -\infty, -1\rangle \cup [-\frac{1}{2}, 0\rangle}$

4. Bez korištenja L'Hospitalovog pravila izračunajte $\lim\limits_{x\to 0} \frac{\operatorname{tg}(x-\pi) - \sin(x-\pi)}{x}$

Rješenje. $\operatorname{tg}(x-\pi) = \frac{\sin(x-\pi)}{\cos(x-\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x}$

$\sin(x-\pi) = -\sin x$, pa je

$$\lim_{x\to 0} \frac{\operatorname{tg}(x-\pi) - \sin(x-\pi)}{x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} + \sin x}{x} = \lim_{x\to 0} \frac{\sin x + \sin x \cdot \cos x}{\cos x \cdot x} =$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{1} \cdot \frac{\overbrace{(1+\cos x)}^{1}}{\underbrace{\cos x}_{1}} = 1 \cdot \frac{1+1}{1} = \boxed{2}$$

5. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{x^2+x}{(x+2)^2}$.

Rješenje. a) $\mathcal{D}(f) = \mathbb{R} \setminus \{-2\}$

b) $\mathcal{N}(f) = ?$ $\quad x^2 + x = 0 \Rightarrow x_1 = 0,\ x_2 = -1 \Rightarrow \mathcal{N}(f) = \{-1, 0\}$

c) asimptote: vertikalna: $x = -2$

horizontalna: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2+x}{x^2+4x+4} \quad /:x =$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1+\frac{4}{x}+\frac{4}{x^2}} = 1 \Rightarrow \underline{y = 1}$$

kosa: nema

d) $$f'(x) = \frac{(2x+1)\cdot(x+2)^{\cancel{2}} - (x^2+x)\cdot 2\cancel{(x+2)}}{(x+2)^{\cancel{4}3}} = \frac{2\cancel{x^2}+5x+2-2\cancel{x^2}-2x}{(x+2)^3} =$$

$$= \frac{3x+2}{(x+2)^3} = 0 \Rightarrow \underline{x = -\tfrac{2}{3}}$$

e) $$f''(x) = \frac{3(x+2)^{\cancel{3}} - (3x+2)\cdot 3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}4}} = \frac{3x+\cancel{6}-9x-\cancel{6}}{(x+2)^4} = \frac{-6x}{(x+2)^4}$$

$$f''\left(-\tfrac{2}{3}\right) = \frac{\overbrace{-6\cdot\left(-\tfrac{2}{3}\right)}^{>0}}{\underbrace{\left(-\tfrac{2}{3}+2\right)^4}_{>0}} > 0 \Rightarrow \left(-\tfrac{2}{3}, f\left(-\tfrac{2}{3}\right)\right) \text{ LOK. MIN.}$$

$$f\left(-\tfrac{2}{3}\right) = \frac{\left(-\frac{2}{3}\right)^2 - \frac{2}{3}}{\left(-\frac{2}{3}+2\right)^2} = \frac{\frac{4}{9}-\frac{2}{3}}{\frac{16}{9}} = \frac{-2}{16} = -\frac{1}{8} \Rightarrow \underline{\left(-\tfrac{2}{3}, -\tfrac{1}{8}\right) \text{ LOK. MIN.}}$$

f) rast: $f'(x) > 0 \Rightarrow \dfrac{3x+2}{\underbrace{(x+2)^3}_{\underbrace{(x+2)^2}_{>0}\cdot(x+2)}} > 0 \Leftrightarrow \dfrac{3x+2}{x+2} > 0$

a) $\left.\begin{array}{l}3x+2>0\\ x+2>0\end{array}\right\}\ x > -\frac{2}{3}$

b) $\left.\begin{array}{l}3x+2<0\\ x+2<0\end{array}\right\}\ x < -2$

$\Rightarrow$ rast: $x \in \langle -\infty, -2\rangle \cup \langle -\frac{2}{3}, \infty\rangle$

$\Rightarrow$ pad: $x \in \langle -2, -\frac{2}{3}\rangle$

g) tok:

	$\langle -\infty, -2\rangle$	-2	$\langle -2, -\frac{2}{3}\rangle$	$-2/3$	$\langle -\frac{2}{3}, \infty\rangle$
f'	↗	×	↘	0	↗
f	+	×	−	MIN	+

h) graf: