

# MATEMATIKA 1

## PISMENI ISPITI 2004. - RJEŠENJA

15. svibnja

23. lipnja

6. srpnja

13. srpnja

9. rujna

23. rujna

1. listopada

11. prosinca

Zadatak. Odredite sve prirodne brojeve  $a$  takve da je domena funkcije  $f(x) = \log_a(x^2 + ax + 3)$  čitav skup  $\mathbb{R}$ .

Rješenje. uvjeti:  $a > 0, a \neq 1$  (zbog baze log. funkcije)  
 $x^2 + ax + 3 > 0$  (zbog argumenta log. funkcije)

Želimo da su uvjeti ovi ispunjeni svim  $x \in \mathbb{R}$ .

Da bi  $x^2 + ax + 3 > 0$  bio cjeli  $\mathbb{R}$ , mora vrijediti  $g(x) > 0$  za sve  $x \in \mathbb{R}$ , gdje je  $g(x) = x^2 + ax + 3$ , a to će biti zadovoljeno ako i samo ako jednačina  $g(x) = 0$  nema realnih rješenja, tj. ako i samo ako

je  $D = a^2 - 4 \cdot 3 \cdot 1 = a^2 - 12 < 0 \Rightarrow a^2 < 12 \Rightarrow |a| < 2\sqrt{3}$ . Sada iz prvog uvjeta ( $a > 0, a \neq 1$ ) sledi  $a \in (2, 3)$ .

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow -2} \frac{\arcsin(x)}{x+2}$ .

Rješenje. supst.  $x+2=t \Rightarrow t \rightarrow 0$   

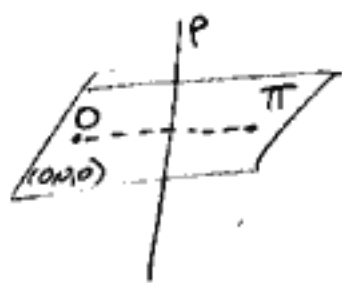
$$\lim_{t \rightarrow 0} \frac{\arcsin(t-2)}{t} = \lim_{t \rightarrow 0} \frac{\sin(\pi t - 2\pi)}{t \cdot \cos(\pi t - 2\pi)} =$$

$$= \lim_{t \rightarrow 0} \frac{\sin \pi t \cdot \cos 2\pi - \cos \pi t \cdot \sin 2\pi}{t (\cos \pi t \cdot \cos 2\pi + \sin \pi t \cdot \sin 2\pi)} =$$

$$= \lim_{t \rightarrow 0} \frac{\sin \pi t}{t \cdot \cos \pi t} = \lim_{t \rightarrow 0} \frac{\frac{\sin \pi t}{\pi t} \cdot \pi}{1 \cdot \cos \pi t} = \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

Zadatak. Odredite tačku simetričnu ishodištu s obzirom na pravac dan jednadžbom  $\frac{x-5}{1} = \frac{y}{2} = \frac{z+3}{-3}$ .

Rješenje. Paralelnu ravninu  $\pi$  kroz  $(0,0,0)$  osovitu na pravac  $\Rightarrow$   
 $\vec{n} = \vec{i} + 2\vec{j} - 3\vec{k} \Rightarrow$  jednačina ravnine je  $x + 2y - 3z = 0$ .



Presjek pravca i ravnine tražimo iz parametarskog oblika jednadžbe pravca  $x = t+5, y = 2t, z = -3t-3$  uvrštavanjem u jednačinu ravnine:

$$t + 5 + 2t + 9t + 9 = 0$$

$$14t = -14 \Rightarrow t = -1 \Rightarrow \text{dobili smo tačku } 0!$$

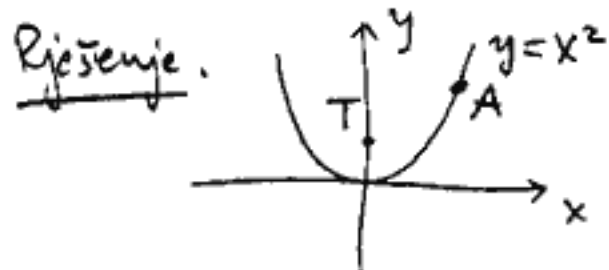
$$O'(4, -2, 0).$$

Da bismo našli tačku  $O''$  simetričnu  $O(0, 0, 0)$ , koristimo identitet

$$\vec{OO''} = 2\vec{OO'}, \text{ gdje } O'' = (x, y, z):$$

$$\rightarrow x\vec{i} + y\vec{j} + z\vec{k} = 2(4\vec{i} - 2\vec{j}) = 8\vec{i} - 4\vec{j} \Rightarrow \left. \begin{matrix} x=8 \\ y=-4 \\ z=0 \end{matrix} \right\} \Rightarrow \boxed{O'' = (8, -4, 0)}$$

Zadatak. Na krivulji  $y = x^2$  nađite tačku najbližu tački  $T(0, 1)$ .



Uzmimo  $A = (x, x^2)$  s krivulje i računamo udaljenost  $d = |\overline{AT}| = \sqrt{(x-0)^2 + (x^2-1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} = \sqrt{x^4 - x^2 + 1}$ .

Minimum udaljenosti se postiže kad funkcija  $f(x) = x^4 - x^2 + 1$

postiže minimum.

$$f'(x) = 4x^3 - 2x = 0$$

$$2x(x^2 - 1) = 0 \Rightarrow x_1 = 0, x_2 = \frac{\sqrt{2}}{2}, x_3 = -\frac{\sqrt{2}}{2}$$

$$f''(x) = 12x^2 - 2$$

$$f''(0) = -2 < 0 \text{ max}$$

$$f''(\pm \frac{\sqrt{2}}{2}) = 12 \cdot \frac{1}{2} - 2 = 4 > 0 \text{ min}$$

$\rightarrow$  imamo dva rješenja:  $A_1(\frac{\sqrt{2}}{2}, \frac{1}{2})$  i  $A_2(-\frac{\sqrt{2}}{2}, \frac{1}{2})$ . Minimalna udaljenost iznosi  $d_{\min} = \frac{\sqrt{3}}{2}$ .

Zadatak. Ispitajte toč i nacrtajte graf funkcije  $f(x) = \frac{x^2 - x}{(x+1)^2}$ .

Rješenje.

1)  $N(f) = \{0, 1\}$   $x^2 - x = 0$

$$x(x-1) = 0 \Rightarrow x_1 = 0, x_2 = 1$$

2)  $D(f) = \mathbb{R} \setminus \{-1\}$

3) asimptote:

V.A.  $x = -1$

H.A.  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 - x}{x^2 + 2x + 1} = 1$

$\Rightarrow y = 1$

K.A. nema

4)  $f'(x) = \frac{(2x-1)(x+1)^2 - (x^2-x) \cdot 2(x+1)}{(x+1)^4} =$

$$= \frac{(x+1)(2x^2+2x-x-1-2x^2+2x)}{(x+1)^4} = \frac{3x-1}{(x+1)^3} = 0 \Rightarrow \underline{x = \frac{1}{3}}$$

kandidat za ekstrem

$$f''(x) = \frac{3(x+1)^3 - (3x-1) \cdot 3(x+1)^2}{(x+1)^6} = \frac{3(x+1)^2(x+1-3x+1)}{(x+1)^6} = \frac{6-6x}{(x+1)^4}$$

$$f''\left(\frac{1}{3}\right) = \frac{6-6 \cdot \frac{1}{3}}{\left(\frac{1}{3}+1\right)^4} > 0 \Rightarrow \text{minimum}; \text{ iznosi } f\left(\frac{1}{3}\right) = \frac{\frac{1}{3}-1}{\left(\frac{1}{3}+1\right)^3} = \underline{\underline{-\frac{1}{8}}}$$

5)  $f'(x) > 0 \Rightarrow \frac{3x-1}{(x+1)^3} = \frac{3x-1}{\underbrace{(x+1)^2}_{>0} \cdot (x+1)} > 0 \Rightarrow$

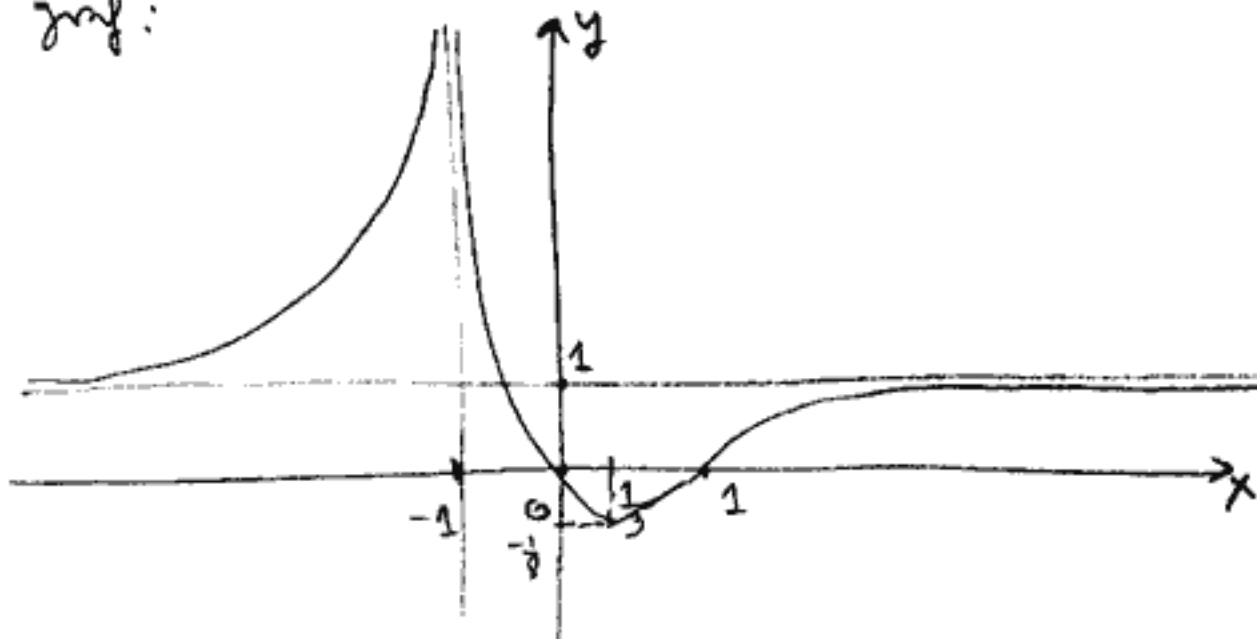
a) $3x-1 > 0$	b) $3x-1 < 0$
$x+1 > 0$	$x+1 < 0$
$x > \frac{1}{3}$	$x < \frac{1}{3}$
$x > -1$	$x < -1$
<u><math>x \in \left(\frac{1}{3}, \infty\right)</math></u>	<u><math>x \in (-\infty, -1)</math></u>

$\Rightarrow$  f raste na  $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$ , a pada na  $\left(-1, \frac{1}{3}\right)$

6)

	$-\infty$	$-1$	$\frac{1}{3}$	$\infty$
	$(-\infty, -1)$	$(-1, \frac{1}{3})$	$(\frac{1}{3}, \infty)$	
$f'$	+	-	0	+
$f$	$\nearrow$	$\searrow$	MM	$\nearrow$

7) graf:



Zadatak. Odredite sjecište pravaca  $P_1 \dots$   $\begin{cases} 2x - 3y + z = 1 \\ x - z = -4 \end{cases}$

$$P_2 \dots \begin{cases} x - 5y + 3z = 8 \\ 2x + y - z = -5 \end{cases}$$

i napišite jednačinu pravca koji je okomit na  $P_1$  i  $P_2$ , a prolazi sjecištem tih pravaca.

Rješenje. Odaberimo tri od ove četiri jednačine da nactemo presječnu tačku - jednako rješenje sustava je  $(-1, 0, 3)$ , i to je tražena tačka (uvrstavanje u četvrtu jednačinu pokazuje da se  $P_1$  i  $P_2$  sijeku).

Kroz tačku  $(-1, 0, 3)$  treba povući pravac okomit na  $P_1$  i  $P_2$ , što znači da za vektor smjera  $\vec{s}$  tog pravca vrijedi  $\vec{s} = \vec{s}_1 \times \vec{s}_2$ ,

gdje su  $\vec{s}_1$  i  $\vec{s}_2$  vektori smjera pravaca  $P_1$  i  $P_2$ .

$$\vec{s}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 3\vec{i} + 3\vec{j} + 3\vec{k}, \quad \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & 3 \\ 2 & 1 & -1 \end{vmatrix} = 2\vec{i} + 7\vec{j} + 11\vec{k}$$

$$\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 3 \\ 2 & 7 & 11 \end{vmatrix} = 12\vec{i} - 27\vec{j} + 6\vec{k}$$

$$\Rightarrow \text{jednačina traženog pravca je } \frac{x+1}{12} = \frac{y}{-27} = \frac{z-3}{15}$$

Zadatak. Odredite domenu funkcije  $f(x) = \log_x(4^x - 6 \cdot 2^x + 8)$ .

Rješenje.  $x > 0, x \neq 1$  (zbog baze log funkcije)

$$4^x - 6 \cdot 2^x + 8 > 0 \quad (\text{zbog argumenta log funkcije})$$

$$(2^x)^2 - 6 \cdot 2^x + 8 > 0, \quad 2^x = t$$

$$t^2 - 6t + 8 > 0 \Rightarrow t_1 < 2 \text{ ili } t_2 > 4$$

$$2^x < 2 \text{ ili } 2^x > 4 \Rightarrow x < 1 \text{ ili } x > 2$$

$$\Rightarrow \boxed{D(f) = \langle 0, 1 \rangle \cup \langle 2, \infty \rangle}$$

Zadatak. Bez upotrebe L'Hospitalnog pravila izračunajte  $\lim_{x \rightarrow 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{1-\sqrt{1-x}}$

Rješenje.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{1-\sqrt{1-x}} \cdot \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \cdot \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}} =$$

$$= \lim_{x \rightarrow 0} \frac{1+\sin x - (1-\sin x)}{1 - (1-x)} \cdot \frac{1+\sqrt{1-x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} =$$

$$= \lim_{x \rightarrow 0} \frac{2\sin x}{x} \cdot \frac{1+\sqrt{1-x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} = 2 \cdot 1 \cdot \frac{1+\sqrt{1}}{\sqrt{1+1} + \sqrt{1-1}} = \boxed{2}$$

Zadatak. Dokazite da sve točke krivulje  $y = \sin x$  u koordinatnom sustavu su tangente paralelne s pravcem  $y = x$  imaju  $y$ -koordinatu nula.

Rješenje. Koeficijent smjera pravca  $y = x$  je  $k = 1$ . Da bi u točki  $(x_0, y_0)$  krivulje  $y = \sin x$  tangenta bila paralelna s  $y = x$ , mora koeficijent smjera tangente biti  $k_t = 1$ . No, koeficijent smjera je  $y'(x_0) = \cos x_0 = 1 \Rightarrow x_0 = 2k\pi, k \in \mathbb{Z}$ . Sada je  $y_0 = \sin x_0 = \sin(2k\pi) = 0$ , što je i trebalo dokazati.

Zadatak. Ispitajte točice i nacrtajte graf funkcije  $f(x) = \frac{x}{\ln x}$ .

Rješenje.

- $D(f) = (0, \infty) \setminus \{1\}$  (zbog  $\ln x \neq 0, x > 0$ )
- $N(f) = \emptyset$
- asimptote:  
 horizontalna  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x}{\ln x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} x = \infty$  nema  
 vertikalna  $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x}{\ln x} = \infty \Rightarrow x = 1$  vertikalna

da li  $y = kx + l$   
 $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{1}{\ln x} = 0$  nema

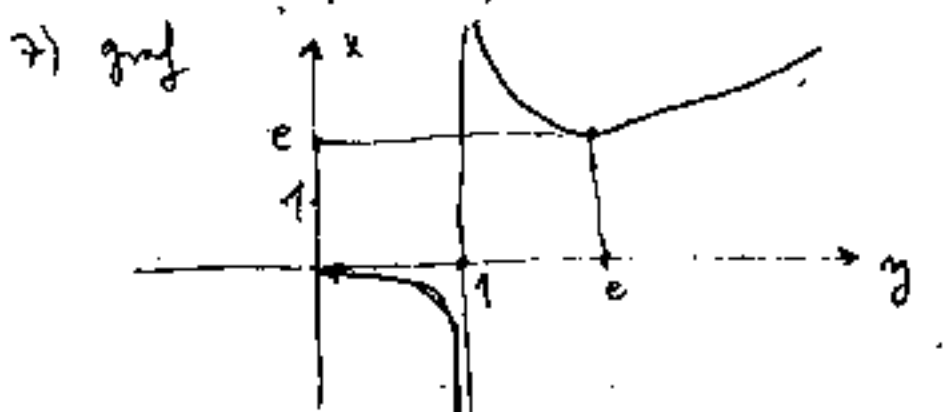
4)  $f'(x) = \frac{\ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{\ln^2 x} = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$   
 $f''(x) = \frac{\frac{1}{x} \cdot \ln^2 x - (\ln x - 1) \cdot 2 \ln x \cdot \frac{1}{x}}{\ln^4 x} = \frac{1}{x} \cdot \frac{\ln x (\ln x - 2 \ln x + 2)}{\ln^3 x} = \frac{2 - \ln x}{x \ln^3 x}$   
 $f''(e) = \frac{2-1}{e \cdot 1} = \frac{1}{e} > 0 \Rightarrow \text{min.}$

$f(e) = \frac{e}{\ln e} = e \Rightarrow (e, e)$  je točka lokalnog minimuma

5)  $f'(x) > 0 \Rightarrow \ln x - 1 > 0 \Rightarrow \ln x > 1 \Rightarrow x > e$  rast  $\Rightarrow x \in (e, \infty) \setminus \{1\}$  pad

6) toz:

	$(0, 1)$	$(1, e)$	$(e, \infty)$
$f'$	-	-	+
$f$	↘	↘	↗



! povećanje u nuli:  
 $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x}{\ln x} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} x = 0$

1. Odredite domenu funkcije  $f(x) = \arcsin(\log_2(\frac{x-1}{x+1}))$ .

Rješenje. uvjeti su: •  $-1 \leq \log_2 \frac{x-1}{x+1} \leq 1$  (zbog arccos)

$$\Rightarrow \frac{1}{2} \leq \frac{x-1}{x+1} \leq 2$$

•  $\frac{x-1}{x+1} > 0$  (zbog  $\log_2$ )

Donji uvjeti ne treba niži rješavati, jer je gornji "jači".

Rješavamo gornji uvjet:

$$\frac{1}{2} \leq \frac{x-1}{x+1} \leq 2$$

1. slučaj  $x+1 > 0$ , tj.  $x > -1 \Rightarrow x \in (-1, \infty)$

$$\Rightarrow x+1 \leq 2x-2 \quad \text{i} \quad x-1 \leq 2x+2$$

$$3 \leq x \quad \text{i} \quad -3 \leq x$$

$\Rightarrow 3 \leq x$ , a rješavamo u skupu  $(-1, \infty) \Rightarrow \underline{y \in [3, \infty)}$

2. slučaj  $x+1 < 0$ , tj.  $x < -1 \Rightarrow x \in (-\infty, -1)$

$$\Rightarrow x+1 \geq 2x-2 \quad \text{i} \quad x-1 \geq 2x+2$$

$$3 \geq x \quad \text{i} \quad -3 \geq x$$

$\Rightarrow -3 \geq x$ , a rješavamo u skupu  $(-\infty, -1) \Rightarrow \underline{x \in (-\infty, -3]}$

$$\Rightarrow \boxed{D(f) = (-\infty, -3] \cup [3, \infty)}$$

2. Bez upotrebe L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow \infty} x(\sqrt{x^2+1} - x)$ .

Rješenje.  $\lim_{x \rightarrow \infty} x(\sqrt{x^2+1} - x) = \lim_{x \rightarrow \infty} (x \cdot (\sqrt{x^2+1} - x) \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}) =$

$$= \lim_{x \rightarrow \infty} \frac{x \cdot ((\sqrt{x^2+1})^2 - x^2)}{\sqrt{x^2+1} + x} = \lim_{x \rightarrow \infty} \frac{x \cdot x}{\sqrt{x^2+1} + x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{x^2}} + 1} = \boxed{\frac{1}{2}}$$

3. Izračunajte približno arctg 1.05.

Rješenje.  $f(x) = \arctg x, x_0 = 1, \Delta x = 0.05 = \frac{5}{100} = \frac{1}{20}$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x, \quad f'(x) = \frac{1}{1+x^2}$$

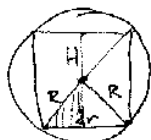
$$\Rightarrow \arctg 1.05 = f(1+0.05) \approx f(1) + f'(1) \cdot \frac{1}{20} = \arctg 1 + \frac{1}{2} \cdot \frac{1}{20} =$$

$$= \frac{\pi}{4} + \frac{1}{40} = \frac{10\pi + 1}{40}$$

$$\Rightarrow \arctg 1.05 \approx \frac{10\pi + 1}{40} \approx 0.803$$

4. Kugli polupjera R upišite valjak maksimalnog volumena.

Rješenje.



$$R, \frac{H}{2}, r \Rightarrow \frac{H^2}{4} + r^2 = R^2$$

$$r^2 = R^2 - \frac{H^2}{4}$$

$$V = r^2 \pi \cdot H = (R^2 - \frac{H^2}{4}) \cdot \pi H$$

$$V' = -\frac{2H}{4} \cdot \pi H + (R^2 - \frac{H^2}{4}) \cdot \pi = 0 \quad / \cdot \frac{4}{\pi}$$

$$-2H^2 + 4R^2 - H^2 = 0$$

$$4R^2 = 3H^2 \Rightarrow H = \frac{2\sqrt{3}}{3} R$$

$$r^2 = R^2 - \frac{1}{4} \cdot \frac{4}{3} R^2 = \frac{2}{3} R^2 \Rightarrow r = \frac{\sqrt{6}}{3} R$$

$$V_{\max} = \frac{2}{3} R^2 \cdot \pi \cdot \frac{2\sqrt{3}}{3} R \Rightarrow V_{\max} = \frac{4\sqrt{3}}{9} R^3 \pi$$

5. Ispitajte toz i nacrtajte graf funkcije  $f(x) = \frac{x-3}{x^2-6x+5}$

Rješenje.

1)  $D(f) = ? \quad x^2 - 6x + 5 = 0 \Rightarrow x_1 = 1, x_2 = 5 \Rightarrow D(f) = \mathbb{R} \setminus \{1, 5\}$

2)  $N(f) = \{3\} \quad x - 3 = 0 \Rightarrow x = 3$

3) asimptote: vertikalna:  $x = 1, x = 5$

horizontalna:  $\lim_{x \rightarrow \infty} \frac{x-3}{x^2-6x+5} = 0 \Rightarrow y = 0$

4)  $f'(x) = \frac{1 \cdot (x^2-6x+5) - (x-3) \cdot (2x-6)}{(x^2-6x+5)^2} = \frac{x^2-6x+5 - 2x^2+6x+6x-18}{(x^2-6x+5)^2}$

$$= \frac{-x^2+6x-13}{(x^2-6x+5)^2} = 0 \Rightarrow x^2-6x+13=0$$

$D = 36 - 52 < 0 \Rightarrow$  nema kandidata za ekstrem

5)  $f''(x) = \frac{(-2x+6)(x^2-6x+5)^2 - (-x^2+6x-13) \cdot 2(x^2-6x+5)(x-6)}{(x^2-6x+5)^4}$  kubna jednačina

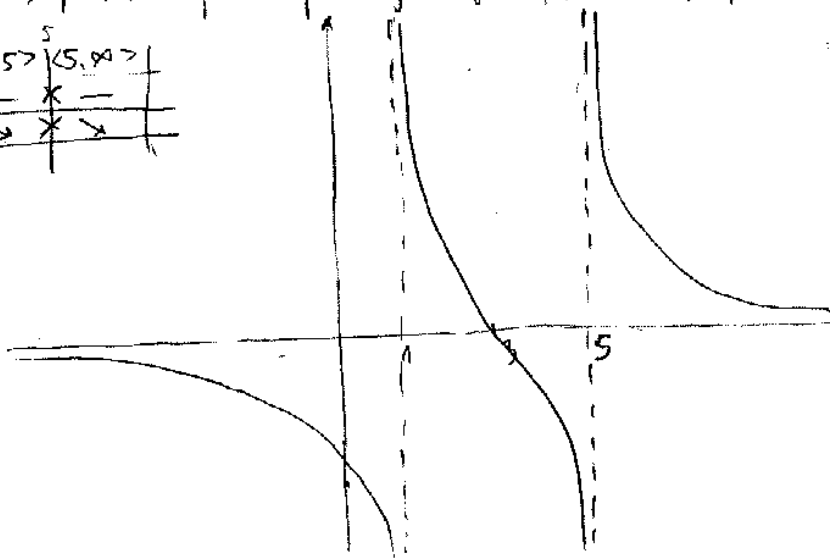
6) rast:  $f'(x) > 0 \Rightarrow -x^2+6x-13 > 0$

$\Rightarrow$  pad na cijelom podmediju definicije, rast nigdje

7)

	$-\infty, 1$	$1, 5$	$5, \infty$
$f'$	-	-	-
$f$	$\nearrow$	$\searrow$	$\searrow$

8) graf:

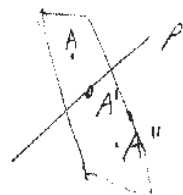


Zadatak. Odredite točku simetričnu točki  $(1, 2, 3)$  s obzirom na pravac

$$P \dots \begin{cases} x+y-z=4 \\ x-y+z=2 \end{cases}$$

Rješenje. Načimo kanonski oblik jednadžbe pravca:

$$\begin{array}{l|l} x+y-z=4 \\ x-y+z=2 \end{array} \quad + \Rightarrow \quad \begin{aligned} x &= 3, z = t, y = 4+t-3 \\ & y = 1+t \end{aligned}$$



$$\Rightarrow \frac{x-3}{0} = \frac{y-1}{1} = \frac{z-0}{1}$$

$$\Rightarrow \vec{s} = \vec{i} + \vec{k}$$

Pomocno ravninu  $\Pi$  koja prolazi točkom  $(1, 2, 3)$  i okomita je na  $P \Rightarrow \vec{n} = \vec{s} = \vec{i} + \vec{k}$ , pa je  $\Pi$  dana s  $0 \cdot (x-1) + 1 \cdot (y-2) + 1 \cdot (z-3)$

$\Rightarrow y+z=5$ . Tražimo presjek  $\Pi \cap P$ :

$$1+t+t=5 \Rightarrow t=2 \Rightarrow A' = (3, 3, 2). \text{ Znamo: } A'' = (x, y, z) : \begin{cases} \frac{x+1}{2} = 3 \\ \frac{y+2}{2} = 3 \\ \frac{z+3}{2} = 2 \end{cases}$$

Zadatak. Dokazite da je domena funkcije  $f(x) = \arcsin \frac{x}{\sqrt{x^2+1}}$  cijeli  $\mathbb{R}$ .

Rješenje. Jedini uvjet je  $-1 \leq \frac{x}{\sqrt{x^2+1}} \leq 1$ , tj.  $\frac{|x|}{\sqrt{x^2+1}} \leq 1 \Leftrightarrow |x| \leq \sqrt{x^2+1}$   
 $\Leftrightarrow x^2 \leq x^2+1$ , što je uvijek istina  $\Rightarrow D(f) = \mathbb{R}$ .

Zadatak. Bez upotrebe L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow 1} \frac{\frac{1}{3} \pi(x)}{x-1}$ .

Rješenje. Supst.  $x-1=t$

$$\begin{aligned} \Rightarrow \lim_{t \rightarrow 0} \frac{\frac{1}{3} \pi(t+1)}{t} &= \lim_{t \rightarrow 0} \frac{\frac{1}{3} (\pi t + \pi)}{t} = \lim_{t \rightarrow 0} \frac{\frac{1}{3} \pi t + \frac{1}{3} \pi}{t} = \\ &= \lim_{t \rightarrow 0} \frac{\frac{1}{3} \pi t}{t} = \lim_{t \rightarrow 0} \frac{\sin \pi t}{t} \cdot \frac{1}{\cos \pi t} = \frac{\frac{1}{3} \pi}{1} = \frac{\pi}{3} \end{aligned}$$

Zadatak. Odredite tangente i normale na Enirulju u točkama presjeka Enirulje

$$y = x^3 - x^2 - 4x + 4 \text{ i } x\text{-osi.}$$

Rješenje.

$$y=0 \Rightarrow x^3 - x^2 - 4x + 4 = 0$$

$$x^2(x-1) - 4(x-1) = 0$$

$$(x^2 - 4)(x-1) = 0 \Rightarrow x_1 = -2, x_2 = 1, x_3 = 2$$

$$y_1 = 0, y_2 = 0, y_3 = 0$$

1. točka (-2, 0) koristimo formulu  $y - y_1 = y'(x_1)(x - x_1)$  tangenta

$$y - y_1 = \frac{-1}{y'(x_1)}(x - x_1) \text{ normala}$$

$$y' = 3x^2 - 2x - 4 \Rightarrow y'(-2) = 12 \Rightarrow y = 12x + 24 \text{ tangenta}$$

$$y = -\frac{1}{12}x - \frac{1}{6} \text{ normala}$$

2. točka (1, 0)  $y'(1) = -3 \Rightarrow y = -3x + 3$  tangenta

$$y = \frac{1}{3}x - \frac{1}{3} \text{ normala}$$

3. točka (2, 0)  $y'(2) = 4 \Rightarrow y = 4x - 8$  tangenta

$$y = -\frac{1}{4}x + \frac{1}{2} \text{ normala}$$

Zadatak. Ispitajte toč i nacrtajte graf funkcije  $f(x) = \frac{x^2 - 4x + 5}{x - 2}$ .

Rješenje.  $D(f) = \mathbb{R} \setminus \{2\}$

•  $N(f) = ?$   $x^2 - 4x + 5 = 0 \Rightarrow$  nema realnih rješenja  $\Rightarrow N(f) = \emptyset$

• asimptote: vertikalna  $x = 2$   
horizontalna  $\lim_{x \rightarrow \infty} f(x) = \infty \rightarrow$  nema

kosa  $y = kx + l$

$$k = \lim_{x \rightarrow \infty} \frac{x^2 - 4x + 5}{x(x-2)} = \frac{1}{1}$$

$$l = \lim_{x \rightarrow \infty} \left( \frac{x^2 - 4x + 5}{x-2} - x \right) = \lim_{x \rightarrow \infty} \frac{-2x + 5}{x-2} = -2$$

$\Rightarrow y = x - 2$  kosa

$$\bullet f'(x) = \frac{(2x-4)(x-2) - (x^2-4x+5)}{(x-2)^2} = \frac{x^2-4x+3}{(x-2)^2} = 0 \Rightarrow x^2-4x+3=0 \Rightarrow x_1=1, x_2=3$$

kandidati  
ekstrem

$$\bullet f''(x) = \frac{(2x-4)(x-2)^{-2} - (x^2-4x+3) \cdot 2(x-2)^{-3}}{(x-2)^4} = \frac{2}{(x-2)^3}$$

$$f''(1) = -2 < 0 \Rightarrow (1, f(1)) = (1, -2) \text{ max}$$

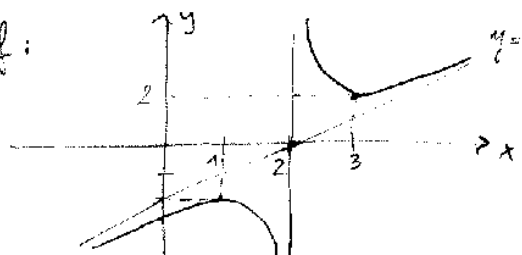
$$f''(3) = 2 > 0 \Rightarrow (3, f(3)) = (3, 2) \text{ min}$$

• rast:  $f'(x) > 0 \Rightarrow \frac{x^2-4x+3}{(x-2)^2} > 0 \Rightarrow x^2-4x+3 > 0 \Rightarrow x \in (-\infty, 1) \cup (3, \infty)$  rast  
 $\Rightarrow x \in (1, 3)$  pad

• toč:

	$-\infty, 1)$	$1, 2)$	$2, 3)$	$3, \infty)$	
$f'$	+	0	-	0	+
$f$		MAX		MIN	

• graf:



MATEMATIKA 1 / 9.9.4. / RJEŠENJA

1. Nadjite jednačinu ravnine koja sadrži pravac  $p_1 \dots \begin{cases} 3x+y-z+5=0 \\ -x+y+z-3=0 \end{cases}$   
 a paralelna je s pravcem  $p_2 \dots \begin{cases} x+y-z+5=0 \\ x+3y=0 \end{cases}$ .

Rješenje. Tražimo  $\Pi$  s vektorom normale  $\vec{n}$  i točkom  $T$ .

$\vec{n} = ?$

$\vec{n} = \vec{s}_1 \times \vec{s}_2$ , gdje je  $\vec{s}_1$  vektor smjera pravca  $p_1$ , a  $\vec{s}_2$  vektor smjera

pravca  $p_2$ .  $\vec{s}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 3 & 0 \end{vmatrix} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \underline{\underline{8\vec{j} + 4\vec{k}}}$

$T = ?$  Možemo uzeti proizvoljnu točku s pravca  $p_1$ . Neka je  $z=0$ , uvrstimo

u  $p_1 \Rightarrow \begin{cases} 3x+y=-5 \\ -x+y=3 \end{cases} \Rightarrow x=-2, y=1 \Rightarrow \underline{\underline{T=(-2, 1, 0)}}$

$\Pi \dots 8(y-1) + 4z = 0 \quad /: 4 \Rightarrow \underline{\underline{2y+z=2}}$

2. Odnadite domenu funkcije  $f(x) = \sqrt{(3\log_{\frac{1}{2}} x - 4\log_{\frac{1}{2}} x + 1)(4x^2 - 1)}$

Rješenje. •  $x > 0$  (zbog log)

•  $(3\log_{\frac{1}{2}} x - 4\log_{\frac{1}{2}} x + 1)(4x^2 - 1) \geq 0$  (zbog  $\sqrt{\quad}$ )

(imamo dvije mogućnosti:

I)  $\begin{cases} 3\log_{\frac{1}{2}} x - 4\log_{\frac{1}{2}} x + 1 \geq 0 \\ 4x^2 - 1 \geq 0 \end{cases}$     II)  $\begin{cases} 3\log_{\frac{1}{2}} x - 4\log_{\frac{1}{2}} x + 1 \leq 0 \\ 4x^2 - 1 \leq 0 \end{cases}$

I)  $4x^2 - 1 \geq 0 \Rightarrow x^2 \geq \frac{1}{4} \Rightarrow |x| \geq \frac{1}{2}$ ; zbog  $x > 0$  je  $\underline{\underline{x \geq \frac{1}{2}}}$

$3\log_{\frac{1}{2}} x - 4\log_{\frac{1}{2}} x + 1 \geq 0$  (nedavno supst.  $\log_{\frac{1}{2}} x = t$ )

$\Rightarrow 3t^2 - 4t + 1 \geq 0 \quad \frac{1}{3} \quad \frac{1}{1} \Rightarrow t \leq \frac{1}{3}$  ili  $t \geq 1$

II)  $4x^2 - 1 \leq 0 \Rightarrow x^2 \leq \frac{1}{4} \Rightarrow |x| \leq \frac{1}{2}$ ;

zbog  $x > 0$  je  $\underline{\underline{x \in (0, \frac{1}{2}]}}$

$3t^2 - 4t + 1 \leq 0 \Rightarrow \frac{1}{3} \leq t \leq 1 \Rightarrow$

$\frac{1}{3} \leq \log_{\frac{1}{2}} x \leq 1 \quad \left(\left(\frac{1}{2}\right)^{-}\right) \Rightarrow$

$\underline{\underline{\frac{1}{2} \geq x \geq \frac{1}{8}}}$      $\Rightarrow \underline{\underline{x \in [\frac{1}{8}, \frac{1}{2}]}}$

$\log_{\frac{1}{2}} x \leq \frac{1}{3} \quad \left(\left(\frac{1}{2}\right)^{-}\right) \Rightarrow x \geq \left(\frac{1}{2}\right)^{\frac{1}{3}}$

$\log_{\frac{1}{2}} x \geq 1 \quad \left(\left(\frac{1}{2}\right)^{-}\right) \Rightarrow x \geq \frac{1}{2^1} = \frac{1}{2}$

$\Rightarrow \underline{\underline{x \in (-\infty, \frac{1}{2}] \cup [\frac{1}{8}, \infty)}}$

$\Rightarrow \underline{\underline{x \geq \frac{1}{8}}}$

Ukupno rješenje pod I) + II) je  $x \in [\frac{1}{2}, \infty)$ , pa je

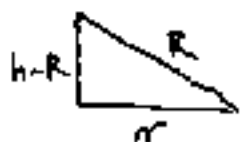
$$D(f) = [\frac{1}{2}, \infty)$$

3. Izračunajte bez L'Hospitalovog pravila  $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$ .

Rješenje.  $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1 - \cos x}{\cos x} \cdot \frac{1}{x^2} =$   
 $= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1 + \cos x}{1 - \cos x} \cdot \frac{1}{x^2} = \frac{1}{2}$

4. U sferi zadane poluprečnika R napišite stavac maksimalnog volumena.

Rješenje.



$$R^2 = (h-R)^2 + r^2$$

$$R^2 = h^2 - 2hR + R^2 + r^2 \Rightarrow r^2 = 2hR - h^2$$

$$V = \frac{1}{3} r^2 \pi \cdot h = \frac{1}{3} \pi \cdot (2hR - h^2) h =$$

$$= \frac{1}{3} \pi (-h^3 + 2h^2 R)$$

$$f'(h) = -3h^2 + 4R = 0 \Rightarrow 3h = 4R \Rightarrow h = \frac{4}{3} R$$

$$r^2 = 2 \cdot \frac{4}{3} R \cdot R - \frac{16}{9} R^2 = \frac{8}{9} R^2 \Rightarrow r = \frac{2\sqrt{2}}{3} R$$

$$V_{max} = \frac{1}{3} \cdot \frac{8}{9} R^2 \cdot \pi \cdot \frac{4}{3} R = \frac{32}{27} R^3 \pi$$

5. Ispitajte tok i nacrtajte graf

funkcije  $f(x) = \frac{2x-3}{(x-2)^2}$ .

Rješenje. 1)  $D(f) = \mathbb{R} \setminus \{2\}$

2)  $x(f) = \{3/2\}$

3) asimptote

V.A.  $x=2$

H.A.  $\lim_{x \rightarrow \infty} f(x) = 0 \Rightarrow y=0$

K.A. nema

4)  $f'(x) = \frac{2(x-2)^2 - (2x-3) \cdot 2(x-2)}{(x-2)^4} = \frac{2x-4-4x+6}{(x-2)^3} = \frac{2-2x}{(x-2)^3} = 0 \Rightarrow x=1$

5)  $f''(x) = \frac{-2(x-2)^3 - (2-2x) \cdot 3(x-2)^2}{(x-2)^6} = \frac{-2x+4-6+6x}{(x-2)^4} = \frac{4x-2}{(x-2)^4} \Rightarrow f''(1) = 2 > 0 \Rightarrow \text{MIN}$   
 $f(1) = -1 \Rightarrow (1, -1)$

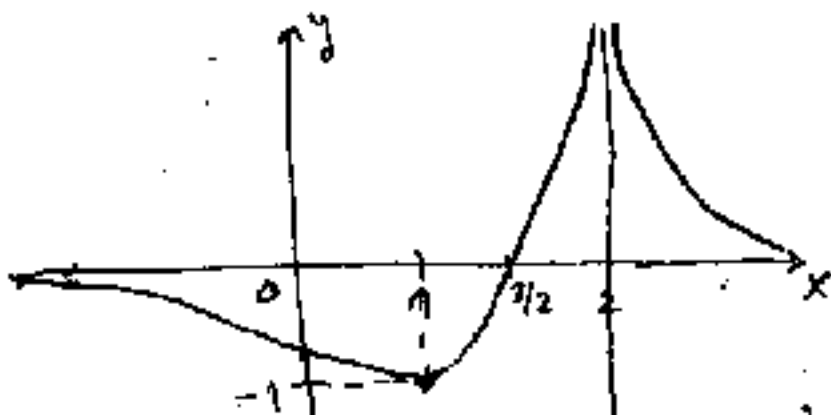
6)  $f(x) > 0 \Rightarrow \frac{2-2x}{(x-2)^2(x-2)} > 0 \Rightarrow$   
 a)  $2-2x > 0 \Rightarrow x < 1$   
 $x-2 > 0 \Rightarrow x > 2$  (nema rješenja)  
 b)  $2-2x < 0 \Rightarrow x > 1$   
 $x-2 < 0 \Rightarrow x < 2$   $\Rightarrow x \in (1, 2)$

$\Rightarrow$  pod:  $x \in (-\infty, 1) \cup (2, \infty)$

7) tok

	$-\infty, 1$	$1, 2$	$2, \infty$
f'	-	+	-
f	$\searrow$ MIN $\nearrow$	$\searrow$ MAX $\nearrow$	

8) graf



1. Odredite  $x \in \mathbb{R}$  tako da vektori  $\vec{a} = (2x-6)\vec{i} + 4\vec{j} - 3\vec{k}$   
 $\vec{b} = (3x-1)\vec{i} + 2\vec{j} + 2\vec{k}$   
 $\vec{c} = (3-8x)\vec{i} + (x-2)\vec{j} - 3x\vec{k}$

budu komplanarni, te u tom slučaju razrite vektor  $\vec{c}$  kao linearnu kombinaciju vektora  $\vec{a}$  i  $\vec{b}$ .

Rješenje:  $\vec{a}, \vec{b}$  i  $\vec{c}$  su komplanarni ako je  $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ , tj. ako je

$$\begin{vmatrix} 2x-6 & 4 & -3 \\ 3x-1 & 2 & 2 \\ 3-8x & x-2 & -3x \end{vmatrix} = (2x-6)(-6x-2x+4) - 4(-9x+3x-6+16x) - 3((3x-1)(x-2)-6+16x) = \dots = 11x^2 - 42x + 12 = 0$$

$\Rightarrow x_1 = \frac{3}{11}, x_2 = 4 \rightarrow 2$  rješenja

1. rješenje:  $x_1 = \frac{3}{11} \Rightarrow \vec{a} = \frac{-60}{11}\vec{i} + 4\vec{j} - 3\vec{k}$

$\vec{b} = \frac{-2}{11}\vec{i} + 2\vec{j} + 2\vec{k}$

$\vec{c} = \frac{9}{11}\vec{i} - \frac{13}{11}\vec{j} - \frac{2}{11}\vec{k}$

$\vec{c} = \alpha \vec{a} + \beta \vec{b} \Rightarrow \frac{9}{11} = \frac{-60}{11}\alpha - \frac{2}{11}\beta$

$\frac{-12}{11} = 4\alpha + 2\beta$

$\frac{-2}{11} = -3\alpha + 2\beta$

$\Rightarrow \alpha = \frac{-10}{77}, \beta = \frac{-93}{154}$

$\Rightarrow \vec{c} = \frac{-10}{77}\vec{a} - \frac{93}{154}\vec{b}$

2. rješenje:  $x_2 = 4 \Rightarrow \vec{a} = 2\vec{i} + 4\vec{j} - 3\vec{k}$

$\vec{b} = 11\vec{i} + 2\vec{j} + 2\vec{k}$

$\vec{c} = -29\vec{i} + 2\vec{j} - 12\vec{k}$

$\vec{c} = \alpha \vec{a} + \beta \vec{b} \Rightarrow -29 = 2\alpha + 11\beta$

$2 = 4\alpha + 2\beta$

$-12 = -3\alpha + 2\beta$

$\alpha = 2, \beta = -3$

$\Rightarrow \vec{c} = 2\vec{a} - 3\vec{b}$

2. Odredite domenu funkcije  $f(x) = \sqrt{\pi^2 - (3\arcsin x)^2}$ .

Rješenje:  $-1 \leq x \leq 1$  (zbog arcsin)

$\pi^2 - (3\arcsin x)^2 \geq 0$  (zbog  $\sqrt{\quad}$ )

$\Rightarrow (3\arcsin x)^2 \leq \pi^2$

$\Rightarrow |3\arcsin x| \leq \pi \Rightarrow -\pi \leq 3\arcsin x \leq \pi \quad | :3$

$-\frac{\pi}{3} \leq \arcsin x \leq \frac{\pi}{3} \Rightarrow$

$-\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2}$

$\Rightarrow \mathcal{D}(f) = [-1, 1] \cap [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}] \Rightarrow \mathcal{D}(f) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$

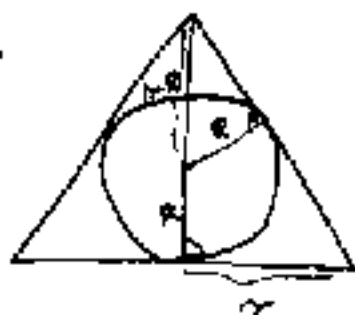
3. Bez korištenja L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow 1} \frac{1-x^2}{\sin \pi x}$ .

Rješenje: Uvedimo supstituciju  $x-1=t$

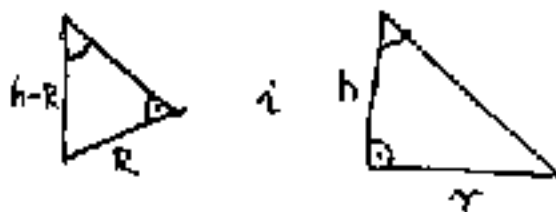
$$\begin{aligned} \Rightarrow \lim_{t \rightarrow 0} \frac{1-(t+1)^2}{\sin(\pi(t+1))} &= \lim_{t \rightarrow 0} \frac{1-t^2-2t-1}{\sin(\pi t + \pi)} = \lim_{t \rightarrow 0} \frac{-t^2-2t}{-\sin \pi t} = \lim_{t \rightarrow 0} \frac{t(t+2)}{\sin \pi t} = 2 \\ &= \lim_{t \rightarrow 0} \frac{\pi t}{\sin \pi t} \cdot \frac{1}{\pi} \cdot (t+2) = \boxed{\frac{2}{\pi}} \end{aligned}$$

Zadatak. Kugli polunijera  $R$  opisite stožac minimalnog volumena.

Rješenje.



Imamo sljedeće trokute:



$$\Rightarrow \frac{R}{r} = \frac{h-R}{\sqrt{h^2+r^2}}$$

$$\begin{aligned} \Rightarrow R\sqrt{h^2+r^2} &= r(h-R) \Rightarrow R^2(h^2+r^2) = r^2(h^2+R^2-2hR) \\ R^2h^2 + R^2r^2 &= r^2h^2 + r^2R^2 - 2hRr^2 \\ R^2h^2 &= r^2(h^2-2hR) \end{aligned}$$

$$\Rightarrow r^2 = \frac{R^2h^2}{h^2-2hR} = \frac{R^2h}{h-2R}$$

$$V_S = \frac{1}{3} \pi r^2 h$$

$$V_S = \frac{1}{3} \pi \cdot h \cdot \frac{R^2h}{h-2R} = \frac{\pi R^2}{3} \cdot \frac{h^2}{h-2R}$$

$$f(h) = \frac{2h(h-2R)-h^2 \cdot 1}{(h-2R)^2} = 0$$

$$h(2h-4R-h) = 0 \quad h \neq 0$$

$$\Rightarrow \boxed{h = 4R}$$

$$\Rightarrow r^2 = \frac{R^2 \cdot 4R}{4R-2R} = \frac{4R^3}{2R} = 2R^2$$

$$\Rightarrow \boxed{r = R\sqrt{2}}$$

$$\boxed{(V_S)_{\min} = \frac{1}{3} \pi R^2 \cdot \frac{(4R)^2}{4R-2R} = \frac{8}{3} \pi R^3}$$

Zadatak. Ispitajte tok i nacrtajte graf funkcije  $f(x) = \frac{x^2-3x}{x-1}$ .

Rješenje.  $D(f) = \mathbb{R} \setminus \{1\}$

$N(f) = \{0, 3\}$

asimptote: vertikalna  $x=1$   
horizontalna  $y=1$

koef:  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x-3}{x-1} = 1 \Rightarrow l = \lim_{x \rightarrow \infty} (f(x)-x) = \lim_{x \rightarrow \infty} \frac{x^2-3x-x^2+x}{x-1} =$

$$= \lim_{x \rightarrow \infty} \frac{-2x}{x-1} = -2 \Rightarrow \underline{y = x-2}$$

ekstremi:  $f'(x) = \frac{(2x-3)(x-1)-(x^2-3x) \cdot 1}{(x-1)^2} = \frac{x^2-2x+3}{(x-1)^2} = 0$

$\Rightarrow$  nema kandidata za ekstreme  
( $D = 4 - 4 \cdot 3 < 0$ !)

test-pod: raste  $f'(x) > 0$

zbog nazivnika  $(x-1)^2$  menabilni

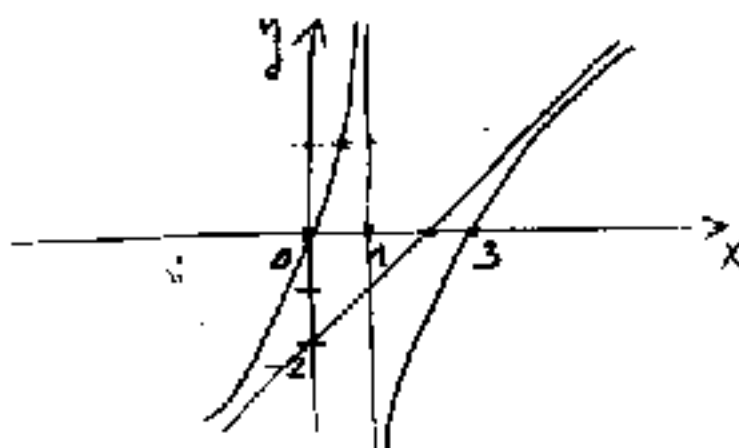
$x^2-2x+3 > 0$ , no to uvijek vrijedi ( $\Delta < 0$ , jer je  $D < 0$ ), pa

$f$  raste na cijeloj domeni

tok:

	$(-\infty, 1)$	$(1, \infty)$
$f'$	+	+
$f$	$\nearrow$	$\nearrow$

graf:



1. Napišite jednadžbu ravnine koja sadrži pravce

$$p_1 \dots \frac{x-3}{2} = \frac{y-2}{1} = \frac{z-1}{2}$$

$$p_2 \dots \frac{x-3}{1} = \frac{y-2}{2} = \frac{z-1}{3}$$

Rješenje.  $p_1 \cap p_2 = \langle (3, 2, 1) \rangle$ , a  $\vec{n} = \vec{s}_1 \times \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\vec{i} - 4\vec{j} + 3\vec{k}$ , pa je

$$-(x-3) - 4(y-2) + 3(z-1) = 0 \quad / \cdot (-1)$$

$$x-3+4y-8-3z+3=0$$

$$\boxed{x+4y-3z=8}$$

2. Odredite domenu funkcije  $f(x) = \arccos \frac{x^2-4}{x^2-1} + \arcsin \frac{5-x^2}{x^2-3}$ .

Rješenje.

a)  $-1 \leq \frac{x^2-4}{x^2-1} \leq 1$  (zbog arccos)

b)  $-1 \leq \frac{5-x^2}{x^2-3} \leq 1$  (zbog arcsin)

a) 1)  $x^2-1 > 0$  množenje nazivnikom ne mijenja predznak:

$$-x^2+1 \leq x^2-4 \leq x^2-1$$

$$\Rightarrow 5 \leq 2x^2 \Rightarrow$$

$$\boxed{|x| \geq \sqrt{\frac{5}{2}}}$$

↑  
kao što vrijedi

2)  $x^2-1 < 0$  množenje nazivnikom mijenja predznak:

$$-x^2+1 \geq x^2-4 \geq x^2-1$$

=> nema rješenja

↑  
kao što ne vrijedi

b)

1)

$$x^2-3 > 0 \Rightarrow$$

$$-x^2+3 \leq 5-x^2 \leq x^2-3 \Rightarrow$$

$$8 \leq 2x^2 \Rightarrow$$

$$x^2 \geq 4 \Rightarrow$$

$$\boxed{|x| \geq 2}$$

↑  
kao što vrijedi

2)

$$x^2-3 < 0 \Rightarrow$$

$$-x^2+3 \geq 5-x^2 \geq x^2-3 \Rightarrow$$

nema rješenja

↑  
kao što ne vrijedi

$$\Rightarrow \boxed{D(f) = \langle -\infty, -2 \rangle \cup [2, \infty \rangle}$$

4. Bez korištenja L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow 0} \left( \frac{x^2 \sin \frac{1}{x}}{\sin x} \right)$ .

Rješenje.

$$\lim_{x \rightarrow 0} \left( \frac{x^2 \sin \frac{1}{x}}{\sin x} \right) = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot x \left( \sin \frac{1}{x} \right)$$

$$\frac{x}{\sin x} \cdot x \left( \sin \frac{1}{x} \right) = \frac{1}{0} \cdot 0 \cdot \left( \sin \frac{1}{x} \right) = \underline{\underline{0}}$$

po apsolutnoj vrijednosti manje od

3. Približno izračunajte  $255.94^{0.25}$ .

Rješenje.

Koristimo formulu  $f(x_0 + \Delta x) \approx \Delta x f'(x_0) + f(x_0)$ , gdje je

$$f(x) = \sqrt[4]{x}, \quad x_0 = 256, \quad \Delta x = -0.06 = \frac{-6}{100} = \frac{-3}{50}$$

$$\Rightarrow 255.94^{0.25} \approx -\frac{3}{50} \cdot \frac{1}{4} (256)^{-3/4} + \sqrt[4]{256} = -\frac{3}{50} \cdot \frac{1}{4} \cdot \frac{1}{4^3} + 4 =$$

$$= 4 - \frac{3}{12800} = \frac{51197}{12800} \approx 3.999765...$$

Zadatak. Ispitajte toč i nacrtajte graf funkcije  $f(x) = \frac{4x-4}{x^2-2x+5}$ .

Rjesenje.  $D(f) = ?$   $x^2-2x+5=0$   $\Delta = 4-20 = -16 < 0 \Rightarrow D(f) = \mathbb{R}$

$$N(f) = \{1\}$$

asimptote: v.a. nema

h.a.  $\lim_{x \rightarrow \infty} \frac{4x-4}{x^2-2x+5} = \lim_{x \rightarrow \infty} \frac{\frac{4}{x} - \frac{4}{x^2}}{1 - \frac{2}{x} + \frac{5}{x^2}} = \frac{0}{1} = 0 \Rightarrow y=0$  v.a.

k.a. nema

$$f'(x) = \frac{4(x^2-2x+5) - (4x-4)(2x-2)}{(x^2-2x+5)^2} = 0 \Rightarrow x^2-2x+5 - (x-1)(2x-2) = 0$$

$$f'(x) = \frac{-4x^2+8x+12}{(x^2-2x+5)^2}$$

$$f''(x) = \frac{(-8x+8)(x^2-2x+5)^2 - (-4x^2+8x+12) \cdot 2(x^2-2x+5) \cdot (2x-2)}{(x^2-2x+5)^4} =$$

$$= \frac{8(-(x-1)(x^2-2x+5) - (x-1)(-2x^2+4x+6))}{(x^2-2x+5)^3}$$

$$= \frac{8(-x^3+2x^2-5x+x^2-2x+5+2x^3-4x^2-6x-2x^2+4x+6)}{(x^2-2x+5)^3}$$

$$= \frac{8(x^3-3x^2-9x+11)}{(x^2-2x+5)^3} \Rightarrow f''(-1) > 0 \Rightarrow (-1, f(-1)) \text{ min}$$

$$f''(3) < 0 \Rightarrow (3, f(3)) \text{ max}$$

$$f(-1) = -1 \Rightarrow (-1, -1) \text{ min}$$

$$f(3) = 1 \Rightarrow (3, 1) \text{ max}$$

rast-pad: rast  $f'(x) > 0$

$$\frac{-4x^2+8x+12}{(x^2-2x+5)^2} > 0$$

$$\Rightarrow -4x^2+8x+12 > 0 \quad | :(-4) \Rightarrow x^2-2x-3 < 0$$

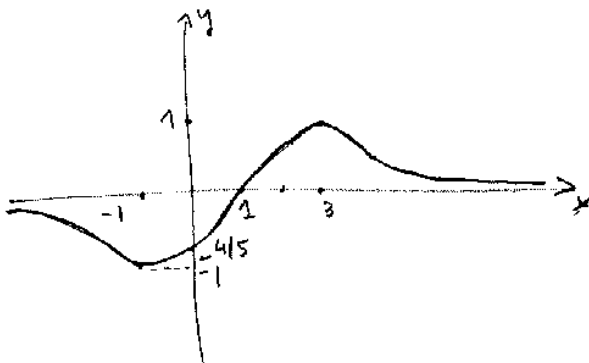
$$\Rightarrow x \in (-1, 3) \text{ rast}$$

$$\Rightarrow \text{pad: } x \in (-\infty, -1) \cup (3, \infty)$$

toč:

	$(-\infty, -1)$	$-1$	$(-1, 3)$	$3$	$(3, \infty)$
$f'$	-	0	+	0	-
$f$		min		max	

graf:



1. Gaussovom metodom eliminacije riješite sustav:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 - x_4 &= 1 \\ x_1 + x_2 - x_3 - 2x_4 &= -8 \\ x_1 - x_2 + 2x_3 + 2x_4 &= 13 \\ 2x_1 + 2x_2 - x_3 - 2x_4 &= -5. \end{aligned}$$

Rješenje.

$$\begin{aligned} &\left[ \begin{array}{cccc|c} 2 & 3 & -1 & -1 & 1 \\ 1 & 1 & -1 & -2 & -8 \\ 1 & -1 & 2 & 2 & 13 \\ 2 & 2 & -1 & -2 & -5 \end{array} \right] \xrightarrow{I \leftrightarrow II} \left[ \begin{array}{cccc|c} 1 & 1 & -1 & -2 & -8 \\ 2 & 3 & -1 & -1 & 1 \\ 1 & -1 & 2 & 2 & 13 \\ 2 & 2 & -1 & -2 & -5 \end{array} \right] \xrightarrow{\substack{I \cdot (-2) + II \rightarrow II \\ I \cdot (-1) + III \rightarrow III \\ I \cdot (-2) + IV \rightarrow IV}} \left[ \begin{array}{cccc|c} 1 & 1 & -1 & -2 & -8 \\ 0 & 1 & 1 & 3 & 17 \\ 0 & -2 & 3 & 4 & 21 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \xrightarrow{\substack{II \cdot (-1) + I \rightarrow I \\ II \cdot 2 + III \rightarrow III}} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 1 & 3 & 17 \\ 0 & 0 & 5 & 10 & 55 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \xrightarrow{III \cdot \frac{1}{5} \rightarrow III} \left[ \begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 1 & 3 & 17 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 1 & 3 & 17 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \left. \begin{array}{l} \text{3. i 4. jednačica su jednake, pa} \\ \text{jednu (četvrtu) možemo} \\ \text{pokušati} \end{array} \right\} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 1 & 3 & 17 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \xrightarrow{\substack{III \cdot 2 + I \rightarrow I \\ III \cdot (-1) + II \rightarrow II}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 2 & 11 \end{array} \right] \end{aligned}$$

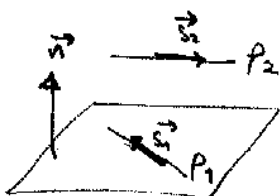
$$\Rightarrow \left. \begin{aligned} x_1 - x_4 &= -3 \\ x_2 + x_4 &= 6 \\ x_3 + 2x_4 &= 11 \end{aligned} \right\} \text{ proglašimo } x_4 \text{ parametrom, tj. } x_4 = t, t \in \mathbb{R}, \text{ pa imamo}$$

$$\begin{aligned} x_1 &= -3 + t \\ x_2 &= 6 - t \\ x_3 &= 11 - 2t \end{aligned}$$

$\Rightarrow$  sustav ima beskonačno mnogo rješenja:  $(-3+t, 6-t, 11-2t, t), t \in \mathbb{R}$ .

2. Nađite jednačinu ravnine koja sadrži pravac  $p_1 \dots \begin{cases} 3x+y-z+5=0 \\ -x+y+z-3=0 \end{cases}$ , a paralelna je s pravcem  $p_2 \dots \begin{cases} x+y-z+5=0 \\ x+3y=0 \end{cases}$ .

Rješenje.



Označimo s  $\Pi$  ravninu koju tražimo te neka je njen vektor normale  $\vec{n}$ . Očito  $\vec{n} \perp \vec{s}_1$  i  $\vec{n} \perp \vec{s}_2$  gdje su  $\vec{s}_1$  i  $\vec{s}_2$  vektori smjera pravca  $p_1$  i  $p_2$ , redom, pa možemo uzeti  $\vec{n} = \vec{s}_1 \times \vec{s}_2$ .

Kako je  $p_1$  zadan kao presjek dvije ravnine, to će biti  $\vec{s}_1 \perp \vec{n}_1, \vec{s}_1 \perp \vec{n}_2$ , gdje su  $\vec{n}_1$  i  $\vec{n}_2$  vektori normale tih ravnina, pa imamo  $\vec{s}_1 = \vec{n}_1 \times \vec{n}_2$ . Analogno je  $\vec{s}_2 = \vec{n}_3 \times \vec{n}_4$ , gdje su  $\vec{n}_3$  i  $\vec{n}_4$  vektori normale ravnina koje određuju  $p_2$ .

$$\Rightarrow \vec{s}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} + 4\vec{k}, \quad \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 3 & 0 \end{vmatrix} = 3\vec{i} - \vec{j} + 2\vec{k}, \text{ pa}$$

$$\text{je } \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = 2\vec{j} + 4\vec{k}.$$

Još nam treba točka  $T_0$  iz  $\Pi$ . Možemo uzeti proizvoljnu točku s pravca  $p_2$  jer  $p_2 \in \Pi$ .

To će biti bilo koja točka  $(x_0, y_0, z_0)$  koja zadovoljava sustav  $\begin{cases} 3x+y-z+5=0 \\ -x+y+z-3=0 \end{cases}$ . Možemo

$$\text{uzeti } z_0=0, \text{ pa imamo } \begin{cases} 3x_0+y_0+5=0 \\ -x_0+y_0-3=0 \end{cases} \Rightarrow 4x_0=-8 \Rightarrow x_0=-2 \Rightarrow y_0=1$$

$$\Rightarrow T_0 = (-2, 1, 0), \text{ pa jednadžba ravnine glasi: } 0 \cdot (x+2) + 8(y-1) + 4(z-0) = 0 / :4$$

$$\boxed{2y+z=2}$$

3. Odredite domenu funkcije  $f(x) = \sqrt{\log_{|x|}(|x|-x)}$ .

Rješenje:  $\bullet$   $|x| > 0, |x| \neq 1 \Rightarrow x \notin \{-1, 1\}$  (zbog baze log funkcije)  
 $x \in \mathbb{R} \setminus \{0\}$

$$\Rightarrow x \in \mathbb{R} \setminus \{-1, 0, 1\}$$

$\bullet$   $|x|-x > 0$  (zbog argumenta log funkcije)

$$\Rightarrow |x| > x \rightarrow \text{ vrijedi za sve } x \in \langle -\infty, 0 \rangle \text{ (za } x \in [0, \infty \rangle \text{ je } |x|=x !)$$

$\bullet$   $\log_{|x|}(|x|-x) \geq 0$  (zbog  $\sqrt{\quad}$  funkcije)

Kako je  $x \in \langle -\infty, 0 \rangle \setminus \{-1\}$ , to gornja nejednakost glasi  $\log_{-x}(-x-x) \geq 0$   
 $\log_{-x}(-2x) \geq 0$

a) ako je  $-x \in \langle 0, 1 \rangle$ , tj.  $x \in \langle -1, 0 \rangle$ , onda imamo  $-2x \leq (-x)^0$   
 $-2x \leq 1$

$$x \geq -\frac{1}{2} \Rightarrow \boxed{x \in [-\frac{1}{2}, 0)}$$

b) ako je  $-x \in \langle 1, \infty \rangle$ , tj.  $x \in \langle -\infty, -1 \rangle$ , onda imamo  $-2x \geq (-x)^0$   
 $-2x \geq 1$

$$x \leq -\frac{1}{2} \Rightarrow \boxed{x \in \langle -\infty, -\frac{1}{2}]}$$

$$\Rightarrow \boxed{D(f) = \langle -\infty, -1 \rangle \cup [-\frac{1}{2}, 0]}$$

4. Bez konstanta L'Hospitalovog pravila izračunajte  $\lim_{x \rightarrow 0} \frac{\operatorname{tg}(x-\pi) - \sin(x-\pi)}{x}$

Rješenje:  $\operatorname{tg}(x-\pi) = \frac{\sin(x-\pi)}{\cos(x-\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x}$

$\sin(x-\pi) = -\sin x$ , pa je

$$\lim_{x \rightarrow 0} \frac{\operatorname{tg}(x-\pi) - \sin(x-\pi)}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} + \sin x}{x} = \lim_{x \rightarrow 0} \frac{\sin x + \sin x \cdot \cos x}{\cos x \cdot x} =$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{\sin x} (1 + \cos x)}{x \cdot \cancel{\cos x}} = 1 \cdot \frac{1+1}{1} = \boxed{2}$$

5. Ispitajte tok i nacrtajte graf funkcije  $f(x) = \frac{x^2+x}{(x+2)^2}$ .

Rješenje.

a)  $D(f) = \mathbb{R} \setminus \{-2\}$

b)  $N(f) = ? \quad x^2 + x = 0 \Rightarrow x_1 = 0, x_2 = -1 \rightarrow N(f) = \{-1, 0\}$

c) asimptote: vertikalna:  $x = -2$   
 horizontalna:  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 + x}{x^2 + 4x + 4} \stackrel{L'H}{=} =$   
 $= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{1 + \frac{4}{x} + \frac{4}{x^2}} = 1 \Rightarrow \underline{y = 1}$

Easca: nema

d)  $f'(x) = \frac{(2x+1) \cdot (x+2)^2 - (x^2+x) \cdot 2(x+2)}{(x+2)^4} = \frac{2x^2 + 5x + 2 - 2x^2 - 2x}{(x+2)^3} =$   
 $= \frac{3x+2}{(x+2)^3} = 0 \Rightarrow \underline{x = -\frac{2}{3}}$

e)  $f''(x) = \frac{3(x+2)^2 - (3x+2) \cdot 3(x+2)}{(x+2)^4} = \frac{3x+6-9x-6}{(x+2)^4} = \frac{-6x}{(x+2)^4}$

$f'(-\frac{2}{3}) = \frac{6 \cdot (-\frac{2}{3})^2}{(-\frac{2}{3}+2)^4} > 0 \Rightarrow (-\frac{2}{3}, f(-\frac{2}{3}))$  Lok. MIN.

$f(-\frac{2}{3}) = \frac{(-\frac{2}{3})^2 - \frac{2}{3}}{(-\frac{2}{3}+2)^2} = \frac{\frac{4}{9} - \frac{2}{3}}{\frac{16}{9}} = \frac{4-2}{16} = \frac{-2}{16} = -\frac{1}{8} \Rightarrow (-\frac{2}{3}, -\frac{1}{8})$  Lok. MIN.

f) rast:  $f'(x) > 0 \Rightarrow \frac{3x+2}{(x+2)^3} > 0 \Leftrightarrow \frac{3x+2}{x+2} > 0$   
 $\frac{3x+2}{(x+2)^2 \cdot (x+2)} > 0$   
 a)  $\left. \begin{matrix} 3x+2 > 0 \\ x+2 > 0 \end{matrix} \right\} x > -\frac{2}{3}$   
 b)  $\left. \begin{matrix} 3x+2 < 0 \\ x+2 < 0 \end{matrix} \right\} x < -2$

$\Rightarrow$  rast:  $x \in \langle -\infty, -2 \rangle \cup \langle -\frac{2}{3}, \infty \rangle$

$\Rightarrow$  pad:  $x \in \langle -2, -\frac{2}{3} \rangle$

g) toki:

	$\langle -\infty, -2 \rangle$	$-2$	$\langle -2, -\frac{2}{3} \rangle$	$-\frac{2}{3}$	$\langle -\frac{2}{3}, \infty \rangle$
$f'$	$\nearrow$	$\times$	$\searrow$	$0$	$\nearrow$
$f$	$+$	$\times$	$-$	MIN	$+$

h) graf:

