MATEMATIKA 1

PISMENI ISPITI 2005. - RJEŠENJA

7. veljače
21. veljače
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MATEMATIKA 1 – 7.2.2005. – RJEŠENJA

Zadatak. Nađite ortogonalnu projekciju točke $T(1,1,1)$ na ravninu Π određenu pravcima $p_1 \ldots \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ i $p_2 \ldots \frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{6}$.

Rješenje. p_1 i p_2 su isti pravci: ako p_1 napišemo parametarski kao $p_1 \ldots x=t, y=2t, z=3t, t\in\mathbb{R}$, a p_2 kao $p_2 \ldots x=2s+1, y=4s+2, z=6s+3, s\in\mathbb{R}$, uz $t\to 2s+1$ vidimo da su to isti pravci, pa ravnina Π nije jednoznačno određena. Stoga i rješenje nije jedinstveno. Ako je vektor normale $\vec{n}$ ravnine Π zadan s $\vec{n}=\vec{s}_1\times\vec{v}$, gdje je $\vec{s}_1=\vec{i}+2\vec{j}+3\vec{k}$ vektor smjera pravca p_1 i p_2, a $\vec{v}=v_0\vec{i}+v_1\vec{j}+v_2\vec{k}$ neki drugi vektor, onda je $\vec{n}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&2&3\\v_0&v_1&v_2\end{vmatrix}=$
$=(2v_2-3v_1)\vec{i}-(v_2-3v_0)\vec{j}+(v_1-2v_0)\vec{k}$. Kao točku u Π možemo uzeti $T_0=(0,0,0)$, pa ravnina glasi $(2v_2-3v_1)x-(v_2-3v_0)y+(v_1-2v_0)z=0$.
Iz točke $T(1,1,1)$ provlačimo pravac p okomit na Π $\Rightarrow$ njegov vektor smjera $\vec{s}$ je jednak $\vec{n}$, pa imamo $p \ldots \frac{x-1}{2v_2-3v_1}=\frac{y-1}{v_2-3v_0}=\frac{z-1}{v_1-2v_0}$.
Tražimo presjek $p\cap\Pi$ – to je tražena ort. projekcija T'.
(*) $x=(2v_2-3v_1)t+1,\ y=(v_2-3v_0)t+1,\ z=(v_1-2v_0)t+1$ uvrstimo u jednadžbu ravnine Π: $((2v_2-3v_1)t+1)(2v_2-3v_1)-(v_2-3v_0)((v_2-3v_0)t+1)$
$+(v_1-2v_0)((v_1-2v_0)t+1)=0 \Rightarrow t'=\dfrac{-2v_2+3v_1+v_2-3v_0-v_1+2v_0}{(2v_2-3v_1)^2-(v_2-3v_0)^2+(v_1-2v_0)^2}$

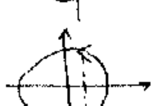
$t'=\dfrac{-v_0+2v_1-v_2}{(2v_2-3v_1)^2-(v_2-3v_0)^2+(v_1-2v_0)^2}$ $\Rightarrow$ taj t' uvršten u param. oblik pravca p (*) daje traženu točku T'.

Zadatak. Odredite domenu funkcije $f(x)=\log_x\left(\sin x-\frac{1}{2}\right)+\log_x\left(\cos x-\frac{1}{2}\right)$.

Rješenje. Uvjeti su:
- $x>0$, $x\neq 1$ (zbog log funkcije – baza)
- $\sin x-\frac{1}{2}>0 \Rightarrow \sin x>\frac{1}{2}$
- $\cos x-\frac{1}{2}>0 \Rightarrow \cos x>\frac{1}{2}$

zbog log funkcije – argument.

$\sin x>\frac{1}{2}$  $\Rightarrow x\in\bigcup_{k\in\mathbb{Z}}\left\langle 2k\pi+\frac{\pi}{6}, 2k\pi+\frac{2\pi}{6}\right\rangle$

$\cos x>\frac{1}{2}$  $\Rightarrow x\in\bigcup_{k\in\mathbb{Z}}\left\langle 2k\pi-\frac{\pi}{3}, 2k\pi+\frac{\pi}{3}\right\rangle$

u presjeku je to $x\in\bigcup_{k\in\mathbb{Z}}\left\langle 2k\pi+\frac{\pi}{6}, 2k\pi+\frac{\pi}{3}\right\rangle$

Zbog $x>0, x\neq 1$ konačno rješenje je $\boxed{\mathcal{D}(f)=\bigcup_{k\in\mathbb{N}}\left\langle 2k\pi+\frac{\pi}{6}, 2k\pi+\frac{\pi}{3}\right\rangle\setminus\{1\}}$.

datak. Bez upotrebe L'Hospitalovog pravila izračunajte $\lim\limits_{x\to1}\frac{1-\sqrt{\cos(x-1)}}{x^2-2x+1}$.

Rješenje. $\lim\limits_{x\to1}\frac{1-\sqrt{\cos(x-1)}}{x^2-2x+1} = \left[\begin{array}{l} t:=x-1\\ \Rightarrow t\to0\end{array}\right] = \lim\limits_{t\to0}\frac{1-\sqrt{\cos t}}{t^2} =$

$$= \lim_{t\to0}\frac{1-\sqrt{\cos t}}{t^2}\cdot\frac{1+\sqrt{\cos t}}{1+\sqrt{\cos t}} = \lim_{t\to0}\frac{1-\cos t}{(1+\sqrt{\cos t})t^2} = \lim_{t\to0}\frac{-2\sin^2\frac t2}{(1+\sqrt{\cos t})t^2} =$$

$$= \lim_{t\to0}\frac{\sin\frac t2\cdot\sin\frac t2}{\underbrace{\tfrac t2}_{1}\cdot\underbrace{\tfrac t2}_{1}\cdot4}\cdot\frac{-2}{1+\underbrace{\sqrt{\cos t}}_{1}} = \frac14\cdot\frac{-2}{2} = \boxed{-\frac14}.$$

datak. U kojoj je točki krivulje $y=x^2+3x+2$ tangenta na tu krivulju paralelna sa simetralom prvog i trećeg kvadranta?

Rješenje. $y'=2x+3$ Tražimo x_0 takav da je $y'(x_0)=1$ (tangenta će tada imati koef. smjera jednak 1, što znači da je paralelna pravcu $y=x$ → to je simetrala prvog i trećeg kvadranta) ⇒ rješavamo jednadžbu

$2x_0+3=1 \Rightarrow x_0=-1 \Rightarrow y_0=1-3+2=0 \Rightarrow$ to je točka $\boxed{(-1,0)}$.

datak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{x^2+x-2}{x+1}$.

Rješenje. $D(f)=\mathbb{R}\setminus\{-1\}$

$N(f)=?$ $x^2+x-2=0 \Rightarrow x_1=1,\ x_2=-2 \Rightarrow N(f)=\{-2,1\}$

asimptote: V.A. $\boxed{x=-1}$

H.A. nema ($\lim\limits_{x\to\infty}f(x)$ ne postoji)

K.A. $y=kx+l$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2+x-2}{x^2+x}=1$$

$$l=\lim_{x\to\infty}(f(x)-x)=\lim_{x\to\infty}\frac{\cancel{x^2}+\cancel{x}-2-\cancel{x^2}-\cancel{x}}{x+1}=0$$

⇒ $\boxed{y=x}$ kosa asimptota

$$f'(x)=\frac{(2x+1)(x+1)-(x^2+x-2)\cdot1}{(x+1)^2}=\frac{x^2+2x+3}{(x+1)^2}=0 \quad D=4-12<0 \Rightarrow \text{nema kandidata za lok. ekstrem}$$

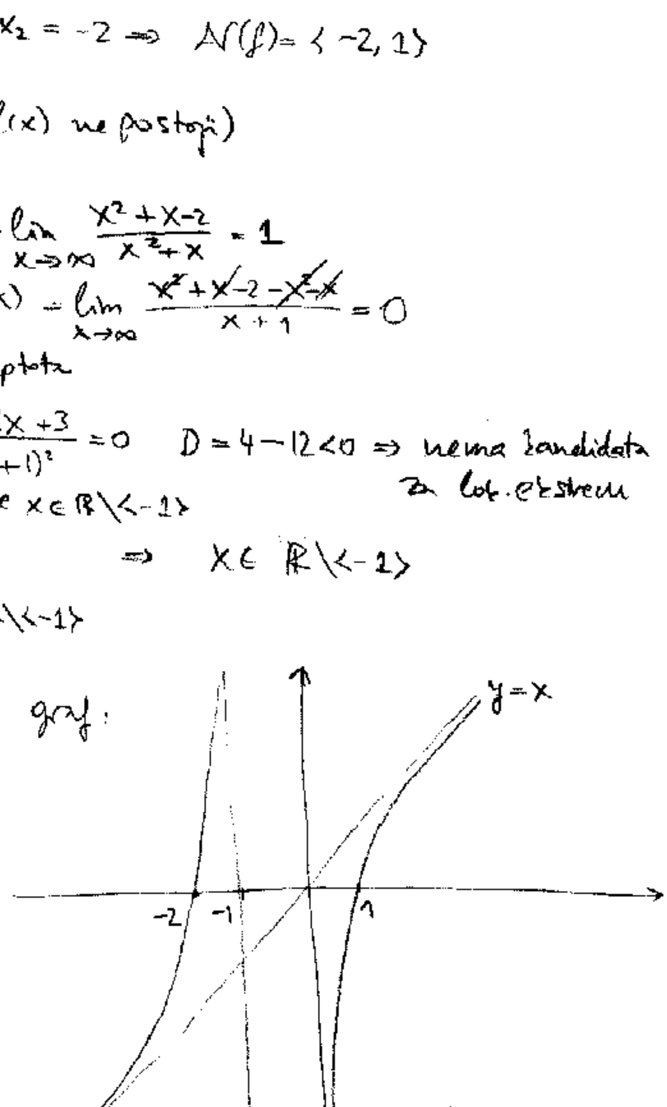
rast: $f'(x)>0 \Rightarrow \frac{\overbrace{x^2+2x+3}^{>0 \text{ za sve } x\in\mathbb{R}\setminus\{-1\}}}{\underbrace{(x+1)^2}_{>0 \text{ za sve } x\in\mathbb{R}\setminus\{-1\}}}>0 \Rightarrow x\in\mathbb{R}\setminus\{-1\}$

pad: $x\in\emptyset$

tok:

	$\langle-\infty,-1\rangle$	-1	$\langle-1,\infty\rangle$
f'	+	x	+
f	↗	x	↗

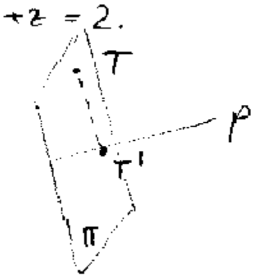
graf:



<u>Zadatak.</u> Nađite ortogonalnu projekciju točke $T(1,1,1)$ na pravac

$$p \dots \begin{cases} x+y-z=1 \\ x-2y+z=2. \end{cases}$$

<u>Rješenje.</u>



Povlačimo pomoćnu ravninu Π okomitu na pravac p, koja prolazi točkom T. Tada će T', točka ort. projekcije, biti na presjeku p i Π.

$\Pi \perp p \Rightarrow \vec{n}$ (vektor normale ravnine Π) $= \vec{s}$ (vektor smjera pravca p). Kako je p zadan kao presjek dvije ravnine (označimo ih s Π_1 i Π_2, s pripadnim vektorima normala $\vec{n_1}$ i $\vec{n_2}$), to je $\vec{n} = \vec{s} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -\vec{i}-2\vec{j}-3\vec{k}$

$$\Rightarrow \Pi \dots \quad -(x-1)-2(y-1)-3(z-1)=0 \quad /\cdot(-1)$$
$$x+2y+3z=6$$

Presjek Π s pravcem p zadanim kao presjek dvije ravnine traži se kao rješenje sustava jednadžbi ravnina Π, Π_1 i Π_2:

$$T' \dots \begin{cases} x+2y+3z=6 \\ x+y-z=1 \\ x-2y+z=2 \end{cases} \Longrightarrow \text{rješenje je jedinstveno i glasi: } x=\tfrac{12}{7},\ y=3/7,\ z=8/7$$

$$\Rightarrow \boxed{T'=\left(\tfrac{12}{7}, \tfrac{3}{7}, \tfrac{8}{7}\right).}$$

<u>Zadatak.</u> Odredite domenu funkcije $f(x)=\ln\left(\frac{2-\sqrt{e^x}}{2+\sqrt{e^x}}\right)$.

<u>Rješenje.</u> Postavljamo uvjete:

I) $\frac{2-\sqrt{e^x}}{2+\sqrt{e^x}} > 0$ (zbog ln)

II) $e^x \geq 0$ (zbog korijena) $\to$ ovaj je uvjet izmah zadovoljen za sve $\boxed{x \in \mathbb{R}}$

Rješavamo I): nazivnik je uvijek pozitivan, pa (da bi razlomak bio pozitivan) mora biti $2-\sqrt{e^x}>0 \Leftrightarrow 2>\sqrt{e^x} \;/()^2 \Rightarrow \; 4>e^x \;/\ln$

$$\Rightarrow \boxed{x<\ln 4}$$

$$\Rightarrow \boxed{D(f)=\langle -\infty, \ln 4\rangle}$$

<u>Zadatak.</u> Bez upotrebe L'Hospitalovog pravila izračunajte $\lim\limits_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{1-\sqrt{1-x}}$

<u>Rješenje.</u>

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{1-\sqrt{1-x}} = \lim_{x\to 0}\left[\frac{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})} \cdot \frac{1+\sqrt{1-x}}{(1-\sqrt{1-x})(1+\sqrt{1-x})}\right] = \lim_{x\to 0}\left[\frac{\cancel{1}-\sin x-\cancel{1}+\sin x}{\sqrt{1+\sin x}+\sqrt{1-\sin x}} \cdot \frac{1+\sqrt{1-x}}{\cancel{1}-\cancel{1}+x}\right] =$$

$$= \lim_{x\to 0} \left[\frac{\cancel{2}\sin x}{x} \cdot \frac{\cancel{1+\sqrt{1-x}}^{\,\cancel{2}}}{\underbrace{\sqrt{1+\sin x}+\sqrt{1-\sin x}}_{\cancel{2}}} \right] = \boxed{2}$$

$\frac{\sin x}{x} \to 1$, $\sin x \to 0$

Zadatak. Dokažite da je za sve vrijednosti negativnog realnog parametra a tangenta na krivulju $y=\ln(ax)$ povučena u točki $(-1, y(-1))$ paralelna s pravcem $x+y=0$.

Rješenje. Tangenta povučena u točki $(-1, y(-1))$ ima koeficijent smjera jednak $y'(-1)$, gdje je $y'(x) = \frac{1}{\cancel{a}x}\cdot\cancel{a} = \frac{1}{x} \Rightarrow y'(-1) = -1$, a to je upravo i koeficijent smjera pravca $x+y=0$, tj. $y=-x$
$\Rightarrow$ oni su paralelni, i to vrijedi za sve $a \in \mathbb{R}^-$.

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{2x+5}{(x+2)^2}$.

Rješenje. $\mathcal{D}(f) = \langle -2 \rangle$

$\mathcal{N}(f) = \{-\frac{5}{2}\}$

asimptote: vertikalna: $x=-2$

horizontalna: $y = \lim_{x\to\infty} f(x) = 0 \Rightarrow \underline{y=0}$

kosa: nema

$$f'(x) = \frac{2(x+2)^{\cancel{2}} - (2x+5)\cdot 2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}} = \frac{2x+4-4x-10}{(x+2)^3} = \frac{-2x-6}{(x+2)^3}$$

$f'(x)=0 \Rightarrow x=-3$ kandidat za lok. ekstrem

$$f''(x) = \frac{-2(x+2)^{\cancel{3}} - (-2x-6)\cdot 3(x+2)^{\cancel{2}}}{(x+2)^{\cancel{6}\,4}} = \frac{-2x-4+6x+18}{(x+2)^4} = \frac{4x+14}{(x+2)^4}$$

$f''(-3) = \frac{2}{(-1)^4} > 0 \Rightarrow (-3, f(-3)) = \underline{(-3,-1)}$ lok. minimum

infleksija: $f''(x)=0 \Rightarrow \underline{x=-\frac{7}{2}}$

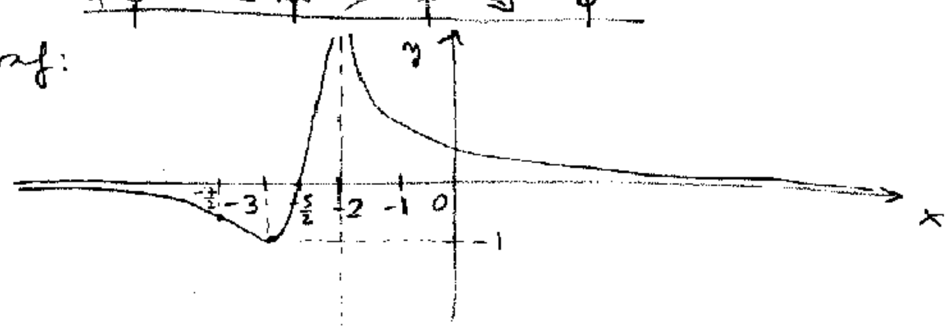
rast: $f'(x)>0 \Rightarrow \frac{-2x-6}{(x+2)^3} = \frac{-2x-6}{\underbrace{(x+2)^2}_{>0 \text{ za sve } x\in\mathbb{R}\setminus\langle -2\rangle}(x+2)} > 0 \Leftrightarrow \frac{2x+6}{x+2} < 0$

$\boxed{x \in \langle -3,-2 \rangle \text{ rast} \Rightarrow x \in \langle -\infty,-3\rangle \cup \langle -2,\infty \rangle \text{ pad}}$

tok:

	$-\infty$	$\langle -\infty,-3\rangle$	-3	$\langle -3,-2\rangle$	-2	$\langle -2,\infty\rangle$	∞
f'	×	−	0	+	×	−	×
f	0	↘	min	↗	×	↘	0

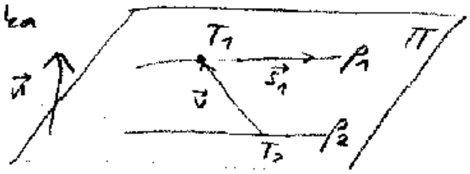
graf:



MATEMATIKA 1 – 12.3.2005. – RJEŠENJA.

<u>adatak.</u> Odredite ortogonalnu projekciju točke $A(-2,3,2)$ na ravninu zadanu pravcima $p_1 \ldots \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$, $p_2 \ldots \frac{x+1}{2} = \frac{y+2}{2} = \frac{z+3}{2}$.

<u>Rješenje.</u> $p_1 \parallel p_2 \rightsquigarrow$ slika



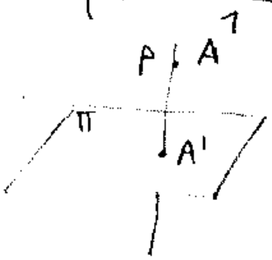
Tražimo $\vec{n}$, vektor normale ravnine π koja sadrži pravce p_1 i p_2. Kako su p_1 i p_2 paralelni, možemo uzeti $\vec{n} = \vec{s}_1 \times \vec{v}$, gdje je $\vec{s}_1$ vektor smjera pravca p_1, a $\vec{v} = \overrightarrow{T_2T_1}$ (T_1 i T_2 su definicijske točke pravaca p_1 i p_2); $T_1 = (0,0,0)$, $T_2 = (-1,-2,-3)$

$$\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k} \Rightarrow \quad \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \vec{i} - 2\vec{j} + \vec{k}$$
$$\vec{s}_1 = \vec{i} + \vec{j} + \vec{k}$$

Kao točku na π možemo uzeti $T_1 = (0,0,0)$, pa jednadžba ravnine π glasi: $\quad x - 2y + z = 0$.

Ortogonalnu projekciju A' točke A nalazimo tako da povučemo pravac p kroz točku A okomit na π $\Rightarrow \vec{s} = \vec{n} = \vec{i} - 2\vec{j} + \vec{k}$ vektor smjera od p

$$\Rightarrow p \ldots \frac{x+2}{1} = \frac{y-3}{-2} = \frac{z-2}{1} = t \Rightarrow \quad \begin{aligned} x &= t-2 \\ y &= -2t+3 \\ z &= t+2 \\ t &\in \mathbb{R} \end{aligned}$$



$A' = p \cap \pi \rightarrow$ uvrstimo param. jednadžbu pravca p u jednadžbu ravnine π:

$$t - 2(-2t+3) + t + 2 = 0$$
$$6t = 6 \Rightarrow t = 1 \Rightarrow \boxed{A' = (-1,1,3)}$$

<u>datak.</u> Odredite domenu funkcije $f(x) = \sqrt{\arcsin(x^2-3)} + \sqrt{\arccos(x^2-3)}$.

<u>Rješenje.</u> Uvjeti: 1) $-1 \le x^2 - 3 \le 1$ (zbog arcsin i arccos funkcije)

2) $\arcsin(x^2-3) \ge 0$ (zbog 1. korijena)

3) $\arccos(x^2-3) \ge 0$ (zbog 2. korijena)

2) $\arcsin(x^2-3) \geq 0 \Rightarrow 0 \leq x^2-3 \leq 1 \,/ +3 \Rightarrow$ 1. uvjet je slabiji, pa ga ne uzimamo u obzir

$3 \leq x^2 \leq 4$

$\sqrt{3} \leq |x| \leq 2 \Rightarrow x \in [-2,-\sqrt{3}] \cup [\sqrt{3},2]$

3) $\arccos(x^2-3) \geq 0 \Rightarrow$ vrijedi za sve $x^2-3 \in \mathcal{D}(\arccos)$ (jer je $\mathcal{R}(\arccos) \subseteq \mathbb{R}^+$)

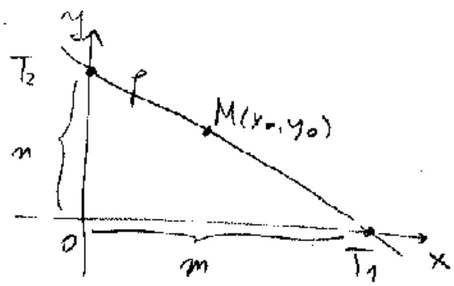
$\Rightarrow \boxed{\mathcal{D}(f) = [-2,-\sqrt{3}] \cup [\sqrt{3},2]}$

Zadatak. Bez upotrebe L'Hospitalovog pravila izračunajte $\lim\limits_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Rješenje. $\lim\limits_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim\limits_{x\to\infty} \frac{\cancel{e^x}(1-\frac{1}{e^{2x}})}{\cancel{e^x}(1+\frac{1}{e^{2x}})} = 1$

Zadatak. U prvom kvadrantu koordinatne ravnine zadana je točka $M(x_0,y_0)$. Povucite kroz tu točku pravac koji s pozitivnim poluosima tvori trokut najmanje površine.

Rješenje.



Pravac p kroz točku M glasi $y-y_0=a(x-x_0)$, gdje je a koef. smjera. Računamo T_1 i T_2, točke presjeka s x-osi i y-osi, redom:

$T_1 \ldots y=0 \Rightarrow \frac{-y_0}{a} = x-x_0 \Rightarrow x = x_0-\frac{y_0}{a}$

$\Rightarrow T_1\left(x_0-\frac{y_0}{a}, 0\right)$

$T_2 \ldots x=0 \Rightarrow y = y_0-ax_0 \Rightarrow T_2(0, y_0-ax_0)$

$P_\Delta = \frac{1}{2}m\cdot n = \frac{1}{2}(y_0-ax_0)\left(x_0-\frac{y_0}{a}\right) \longrightarrow$ to je funkcija po a, tako ćemo je i derivirati

$P_\Delta'(a) = \frac{1}{2}\cdot(-x_0)\left(x_0-\frac{y_0}{a}\right) + \frac{1}{2}(y_0-ax_0)\cdot\frac{+y_0}{a^2} = 0 \,/\cdot 2$

$-x_0^2 + \cancel{\frac{x_0y_0}{a}} + \frac{y_0^2}{a^2} - \cancel{\frac{x_0y_0}{a}} = 0$

$\frac{y_0^2}{a^2} = x_0^2 \Rightarrow a^2 = \frac{y_0^2}{x_0^2} \Rightarrow a = \frac{-y_0}{x_0}$ (mora biti padajući pravac)

$\Rightarrow$ to je pravac $y = a(x-x_0)+y_0$ s koef. smjera $\boxed{a = \frac{-y_0}{x_0}}$:

$y = \frac{-y_0}{x_0}(x-x_0)+y_0 \Rightarrow \boxed{y = \frac{-y_0}{x_0}x + 2y_0}$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{x^2}{2x-5}$.

Rješenje. $\mathcal{D}(f)=\mathbb{R}\setminus\{\frac{5}{2}\}$

$\mathcal{N}(f)=\{0\}$

asimptote: vertikalna $x=\frac{5}{2}$

horizontalna nema

kosa $y=kx+l$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x}{2x-5}=\frac{1}{2}$$

$$l=\lim_{x\to\infty}\left(f(x)-\frac{1}{2}x\right)=\lim_{x\to\infty}\frac{\cancel{x^2}-\cancel{x^2}+\frac{5}{2}x}{2x-5}=\frac{5}{4}$$

$$\Rightarrow y=\frac{1}{2}x+\frac{5}{4}$$

$$f'(x)=\frac{(2x)(2x-5)-x^2\cdot 2}{(2x-5)^2}=\frac{2x^2-10x}{(2x-5)^2}=\frac{2x(x-5)}{(2x-5)^2}=0 \Rightarrow \begin{array}{l} x_1=0 \\ x_2=5 \end{array}$$

kandidati za lok. ekstreme

$$f''(x)=\frac{(4x-10)(2x-5)^{\cancel{2}}-(2x^2-10x)\cdot 2\cancel{(2x-5)}}{(2x-5)^{\cancel{4}3}}=$$

$$=\frac{8x^2-20x-\cancel{20x}+50-4x^2+\cancel{20x}}{(2x-5)^3}=\frac{4x^2-20x+50}{(2x-5)^3}$$

$f''(0)=\frac{50}{-125}<0 \Rightarrow (0,f(0))=(0,0)$ lok. MAX.

$f''(5)=\frac{50}{125}>0 \Rightarrow (5,f(5))=(5,5)$ lok MIN.

rast-pad: rast $f'(x)>0 \Rightarrow \frac{2x(x-5)}{\underbrace{(2x-5)^2}_{\vee}}>0 \Rightarrow 2x(x-5)>0$

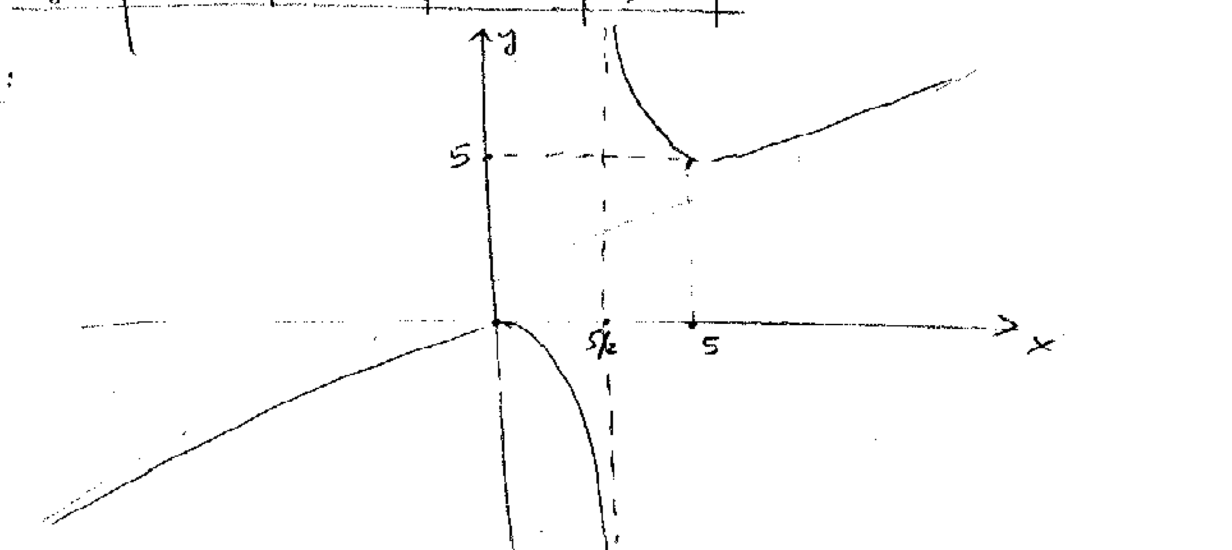
$\Rightarrow x\in\langle-\infty,0\rangle\cup\langle5,\infty\rangle$

$\Rightarrow$ pad $x\in\langle0,5\rangle$

tok:

	$-\infty$	$\langle-\infty,0\rangle$	0	$\langle0,\frac{5}{2}\rangle$	$5/2$	$\langle5/2,5\rangle$	5	$\langle5,\infty\rangle$	∞
f'	×	+	0	−	×	−	0	+	×
f	$-\infty$	↗	MAX	↘	×	↘	MIN	↗	$+\infty$

graf:



MATEMATIKA 1 - 9.4.2005. - RJEŠENJA

1. Napišite jednadžbu pravca koji prolazi točkom $T(1,3,5)$, a okomit je na pravce $p_1 \ldots \begin{cases} x+y-z=1 \\ x-y+z=1 \end{cases}$, $p_2 \ldots \begin{cases} x+y-z=2 \\ -x+y+z=2. \end{cases}$

<u>Rješenje.</u> Označimo sa $\vec{s_1}$ vektor smjera pravca p_1, a sa $\vec{s_2}$ pravca p_2.

Vrijedi $\vec{s_1} = \vec{n_1} \times \vec{n_2}$, $\vec{s_2} = \vec{n_3} \times \vec{n_4}$, gdje su $\vec{n_1}$ i $\vec{n_2}$ vektori normala ravnine koje definiraju p_1, a $\vec{n_3}$ i $\vec{n_4}$ ravnina koje definiraju p_2.

Imamo $\vec{s_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -2\vec{j} - 2\vec{k}$ $\quad \vec{s_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\vec{i} + 2\vec{k}$

Označimo sa $\vec{s}$ vektor smjera pravca p. Kako je $p \perp p_1, p \perp p_2$, to možemo uzeti da je $\vec{s} = \vec{s_1} \times \vec{s_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & -2 \\ 2 & 0 & 2 \end{vmatrix} = -4\vec{i} - 4\vec{j} + 4\vec{k}$, ili $\vec{s} = -\vec{i} - \vec{j} + \vec{k}$

$$\Rightarrow \boxed{p \ldots \frac{x-1}{-1} = \frac{y-3}{-1} = \frac{z-5}{1}}$$

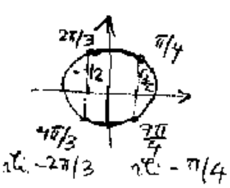
---

2. Odredite domenu funkcije $f(x) = \sqrt{\dfrac{1+2\cos x}{\sqrt{2}-2\cos x}} + \sqrt{9-x^2}$.

<u>Rješenje.</u> Uvjeti: • $9-x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow |x| \leq 3$, tj. $x \in [-3,3]$

• $\dfrac{1+2\cos x}{\sqrt{2}-2\cos x} \geq 0 \rightarrow$ 2 mogućnosti:

1) $\left.\begin{array}{l} 1+2\cos x \geq 0 \\ \sqrt{2}-2\cos x > 0 \end{array}\right\} \Rightarrow \cos x \in \left[-\frac{1}{2}, \frac{\sqrt{2}}{2}\right\rangle$:



Kako je $x \in [-3,3]$, to je

$x \in \left[-\frac{2\pi}{3}, -\frac{\pi}{4}\right\rangle \cup \left\langle \frac{\pi}{4}, \frac{2\pi}{3}\right]$

2) $\left.\begin{array}{l} 1+2\cos x \leq 0 \Rightarrow \cos x \leq -\frac{1}{2} \\ \sqrt{2}-2\cos x < 0 \rightarrow \cos x > \frac{\sqrt{2}}{2} \end{array}\right\}$ nema rješenja

$$\Rightarrow \boxed{\mathcal{D}(f) = \left[-\frac{2\pi}{3}, -\frac{\pi}{4}\right\rangle \cup \left\langle \frac{\pi}{4}, \frac{2\pi}{3}\right]}$$

---

3. Izračunajte bez korištenja L'Hospitalovog pravila $\lim\limits_{x\to 1} (1-x)\operatorname{tg}\frac{\pi x}{2}$.

<u>Rješenje.</u> $\lim\limits_{x\to 1} (1-x)\operatorname{tg}\frac{\pi x}{2} = \left[\begin{array}{c} x-1=t \\ t\to 0 \end{array}\right] = \lim\limits_{t\to 0}\left(-t \dfrac{\sin\frac{\pi}{2}(t+1)}{\cos\frac{\pi}{2}(t+1)}\right) = \lim\limits_{t\to 0}\left(-t \cdot \dfrac{\sin\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)}\right)$

$= \lim_{t\to 0} \left(-t \cdot \frac{\cos\frac{\pi}{2}t}{-\sin\frac{\pi}{2}t}\right) = \lim_{t\to 0}\left(\underbrace{\frac{\frac{\pi}{2}\cancel{t}}{\sin(\frac{\pi}{2}t)}}_{1} \cdot \frac{+2}{\pi}\cdot \underbrace{\cos(\frac{\pi}{2}t)}_{1}\right) = \boxed{+\frac{2}{\pi}}.$

---

4. Za koje vrijednosti $a \in \mathbb{N}$ postoji tangenta na krivulju $y = x^3 + a^2x$ paralelna pravcu $y=16x$?

Rješenje. Svaka tangenta u $(x_0, y(x_0))$ ima koef. smjera $y'(x_0) = 3x_0^2 + 2a$ i ona će biti paralelna ako je taj koeficijent jednak koef. smjera pravca $y=16x \Rightarrow$ mora biti $3x_0^2+2a=16 \Rightarrow 3x_0^2 = 16-2a \Rightarrow x_0^2 = \frac{16-2a}{3}$

$\Rightarrow$ takav $x_0\in\mathbb{R}$ postoji ako $16-2a \geq 0 \Rightarrow 16 \geq 2a \Rightarrow a\le 4 \Rightarrow \boxed{a\in\{1,2,3,4\}}$

---

5. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{3x^4+1}{x^3}$.

Rješenje. $D(f) = \mathbb{R}\setminus\{0\}$

$N(f)=?$ $\quad 3x^4+1>0 \ \forall x\in\mathbb{R} \Rightarrow$ nema rješenja

asimptote: vertikalna: $x=0$

horizontalna: $\lim_{x\to\infty} \frac{3x^4+1}{x^3} = \infty \Rightarrow$ nema

kosa: $y = kx + l \quad k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3x^4+1}{x^4} = 3$

$l = \lim_{x\to\infty} (f(x)-3x) = \lim_{x\to\infty} \frac{\cancel{3x^4}+1-\cancel{3x^4}}{x^3} = 0$ $\Rightarrow y=3x$ kosa

ekstremi

$f'(x)=?$ $\quad f(x)=3x+\frac{1}{x^3}$

$\Rightarrow f'(x) = 3 - \frac{3}{x^4} = 0 \Rightarrow 1-\frac{1}{x^4}=0 \Rightarrow x^4=1 \Rightarrow x^2=1 \Rightarrow x_1=-1, x_1=1$

$f''(x) = \frac{+12}{x^5}$ $\quad f''(1)=12>0 \Rightarrow (1,f(1)) = (1,4)$ MIN

$f''(-1) = -12 < 0 \Rightarrow (-1,f(-1)) = (-1,-4)$ MAX

rast-pad:

$f'(x)>0$ (rast) $\Rightarrow 3-\frac{3}{x^4}>0 \Rightarrow \frac{3(x^4-1)}{\underbrace{x^4}_{>0\ \forall x\in\mathbb{R}\setminus\{0\}}} > 0 \Rightarrow x^4-1>0 \Rightarrow (x^2-1)\underbrace{(x^2+1)}_{>0\ \forall x\in\mathbb{R}}>0 \Rightarrow x^2-1>0$

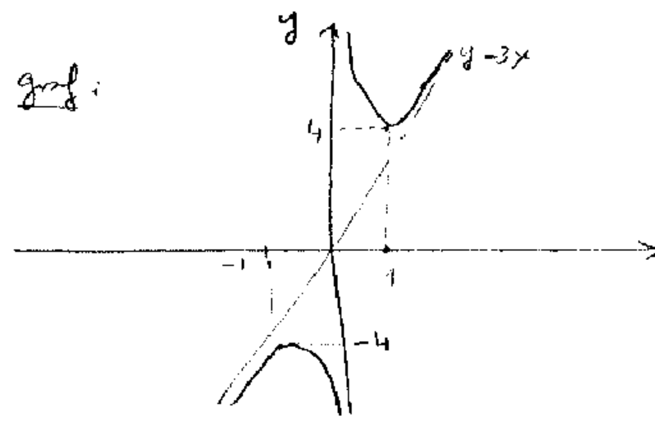
$\Rightarrow$ rast: $x\in\langle-\infty,-1\rangle\cup\langle1,\infty\rangle$

$\Rightarrow$ pad: $x\in\langle-1,1\rangle$

tok:

	$-\infty$	$\langle-\infty,-1\rangle$	-1	$\langle-1,0\rangle$	0	$\langle0,1\rangle$	1	$\langle1,\infty\rangle$	∞
f'	$\times$	+	0	−	$\times$	−	0	+	$\times$
f	$-\infty$	↗	MAX	↘	$\times$	↘	MIN	↗	$+\infty$

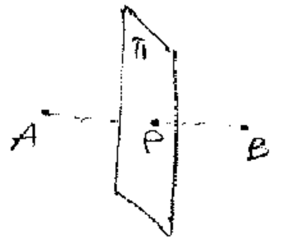
graf:



MATEMATIKA 1 – 14.5.2005. – RJEŠENJA

Zadatak. Nađite jednadžbu ravnine simetralne točkama $A(1,3,0)$ i $B(2,0,3)$.

Rješenje



Tražimo ravninu Π okomitu na pravac p_{AB} koja prolazi ~~kroz~~ točku P polovišta spojnice točaka A i B.

$$P = \left(\frac{1+2}{2}, \frac{3+0}{2}, \frac{0+3}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$$

Kako je $\Pi \perp p_{AB}$, možemo uzeti da je vektor normale $\vec{n}$ ravnine Π jednak vektoru smjera $\vec{s}$ pravca p_{AB}.

$$\vec{n} = \vec{s} = (2-1)\vec{i} + (0-3)\vec{j} + (3-0)\vec{k} = \vec{i} - 3\vec{j} + 3\vec{k}$$

Jednadžba ravnine glasi: $1\cdot\left(x-\frac{3}{2}\right) - 3\left(y-\frac{3}{2}\right) + 3\left(z-\frac{3}{2}\right) = 0$

$$x - \frac{3}{2} - 3y + \cancel{\frac{9}{2}} + 3z - \cancel{\frac{9}{2}} = 0 \;/\cdot 2$$

$$\boxed{2x - 6y + 6z = 3}$$

Zadatak. Odredite domenu funkcije $f(x) = \ln \dfrac{1}{x+\sqrt{x^2-1}}$.

Rješenje. Uvjeti su:

1) $x^2-1 \geq 0$ (zbog korijena)

2) $x+\sqrt{x^2-1} \neq 0$ (zbog nazivnika)

3) $\dfrac{1}{x+\sqrt{x^2-1}} > 0$ (zbog logaritma)

1) $x^2-1 \geq 0 \Leftrightarrow x^2 \geq 1 \Leftrightarrow |x| \geq 1 \Leftrightarrow x \in \langle -\infty, -1] \cup [1, \infty\rangle$

2) ne treba rješavati, jer 3. uvjet ionako zahtijeva drugi

3) $\dfrac{1}{x+\sqrt{x^2-1}} > 0 \rightarrow x + \sqrt{x^2-1} > 0$

$$\sqrt{x^2-1} > -x$$

Imamo 2 slučaja:

a) $-x \geq 0$, tj. $x \leq 0 \rightarrow$ možemo kvadrirati nejednadžbu, pa imamo:
$\cancel{x^2} - 1 > \cancel{x^2} \Rightarrow -1 > 0$, što očito ne vrijedi $\Rightarrow x \in \emptyset$

b) $-x < 0$, tj. $x > 0 \Rightarrow$ u ovom slučaju nejednakost sigurno vrijedi, jer imamo $\sqrt{x^2-1} \geq 0 > -x$

$\Rightarrow$ rješenje 3. uvjeta je $x \in \langle 0, \infty\rangle$ što zajedno s 1. uvjetom daje

$$\boxed{\mathcal{D}(f) = [1, \infty\rangle}$$

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte $\lim_{x\to 1} \frac{\tan \pi x}{x-1}$.

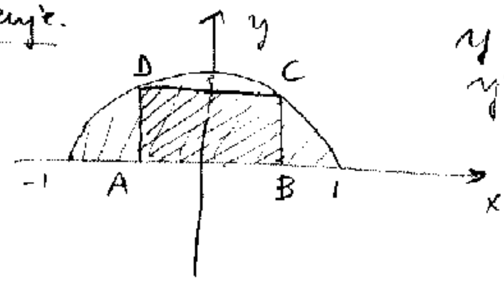
Rješenje. $\lim_{x\to 1} \frac{\tan \pi x}{x-1} = \begin{bmatrix} x-1=t \\ t\to 0 \end{bmatrix} = \lim_{t\to 0} \frac{\tan \pi(t+1)}{t} =$

$= \lim_{t\to 0} \frac{\tan(\pi t+\pi)}{t} = \lim_{t\to 0} \frac{1}{t}\cdot\frac{\sin(\pi t+\pi)}{\cos(\pi t+\pi)} = \lim_{t\to 0} \frac{1}{t}\cdot\frac{\sin \pi t}{\cos \pi t} =$

$= \lim_{t\to 0} \underbrace{\frac{\sin \pi t}{\pi t}}_{1}\cdot \pi \cdot \frac{1}{\underbrace{\cos \pi t}_{1}} = \boxed{\pi}$.

Zadatak. U područje $0 \le y \le \sqrt{1-x^2}$ upišite pravokutnik maksimalne površine.

Rješenje.



$y = \sqrt{1-x^2}$

$y^2 = 1-x^2 \Rightarrow x^2+y^2=1 \Rightarrow$ gornja krivulja predstavlja gornju polukružnicu $S(0,0), r=1$

Zbog simetričnosti slike je očito da su točke pravokutnika na x-osi $A(-a,0), B(a,0)$, za neki $a>0$. Stoga je $C(a,\sqrt{1-a^2})$, $D(-a,\sqrt{1-(-a)^2})$, pa površina pravokutnika glasi $P = |AB|\cdot|BC| = 2a\cdot\sqrt{1-a^2}$. Vidimo da je P funkcija jedne varijable a, a njen maksimum dobivamo deriviranjem:

$P'(a) = 2\sqrt{1-a^2} + \cancel{2}a\cdot\frac{1}{\cancel{2}\sqrt{1-a^2}}\cdot(-2a) = 0 \quad / \cdot\sqrt{1-a^2}$

$2(1-a^2)-2a^2=0 \Rightarrow 2=4a^2 \Rightarrow a^2=\frac{1}{2} \Rightarrow a=\frac{\sqrt2}{2}$

Računanjem $P''(a)$ i uvrštavanjem $a=\frac{\sqrt2}{2}$ vidimo da je $P''(\frac{\sqrt2}{2})<0$, tj. u $a=\frac{\sqrt2}{2}$ P postiže lokalni maksimum.

$\Rightarrow$ Radi se o točkama s vrhovima $A(-\frac{\sqrt2}{2},0), B(\frac{\sqrt2}{2},0), C(\frac{\sqrt2}{2},\frac{\sqrt2}{2}), D(-\frac{\sqrt2}{2},\frac{\sqrt2}{2})$ čije su stranice duljina $\sqrt2$ i $\frac{\sqrt2}{2}$, a površina je $P_{max} = \sqrt2\cdot\frac{\sqrt2}{2} = 1$.

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{2x-1}{(x-1)^2}$.

Rješenje.
1) $\mathcal{D}(f) = \mathbb{R}\setminus\{1\}$
2) $\mathcal{N}(f) = \{\frac{1}{2}\}$
3) asimptote:

V.A. $x = 1$

H.A. $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{2x-1}{(x-1)^2} \overset{L'H}{=} \lim\limits_{x\to\pm\infty} \frac{2-\cancel{\frac{1}{x}}^{\to 0}}{x-2+1} = 0$

$\Rightarrow y=0$ je horizontalna asimptota

K.A. nema

4) ekstremi: $f'(x) = \frac{2\cdot(x-1)^{\cancel{2}} - (2x-1)\cdot 2\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} = \frac{2x-2-4x+2}{(x-1)^3} = \frac{-2x}{(x-1)^3} = 0$

$\Rightarrow$ $x=0$ točka kandidat za lok. ekstrem

5) $f''(x) = \frac{-2\cdot(x-1)^{\cancel{3}} + 2x\cdot 3\cancel{(x-1)}^2}{(x-1)^{\cancel{6}}} = \frac{-2x+2+6x}{(x-1)^4} = \frac{4x+2}{(x-1)^4}$

$f''(0) = \frac{2}{(-1)^4} > 0 \Rightarrow (0, f(0))$ lok. minimum

$f(0) = \frac{-1}{(-1)^2} = -1 \Rightarrow \underline{(0,-1)}$ točka lok. minimuma

točka infleksije: $f''(x)=0 \Rightarrow 4x+2=0 \Rightarrow x=-\frac{1}{2}$, $f(-\frac{1}{2}) = \frac{2\cdot\frac{-1}{2}-1}{(-\frac{1}{2}-1)^2} = \frac{-2}{\frac{9}{4}}$

$\Rightarrow \underline{(-\frac{1}{2}, \frac{-8}{9})}$ točka infleksije $f(-\frac{1}{2}) = \frac{-8}{9}$

6) tok:

rast $\leftrightarrow f'(x) > 0 \Rightarrow \frac{-2x}{(x-1)^3} > 0 \Rightarrow \frac{-2x}{x-1}\cdot\underbrace{\frac{1}{(x-1)^2}}_{>0\ \forall x\in\mathbb{R}\setminus\{1\}} > 0 \Rightarrow \frac{x}{x-1} < 0$

a) $x<0$
$x-1>0$
nemoguće

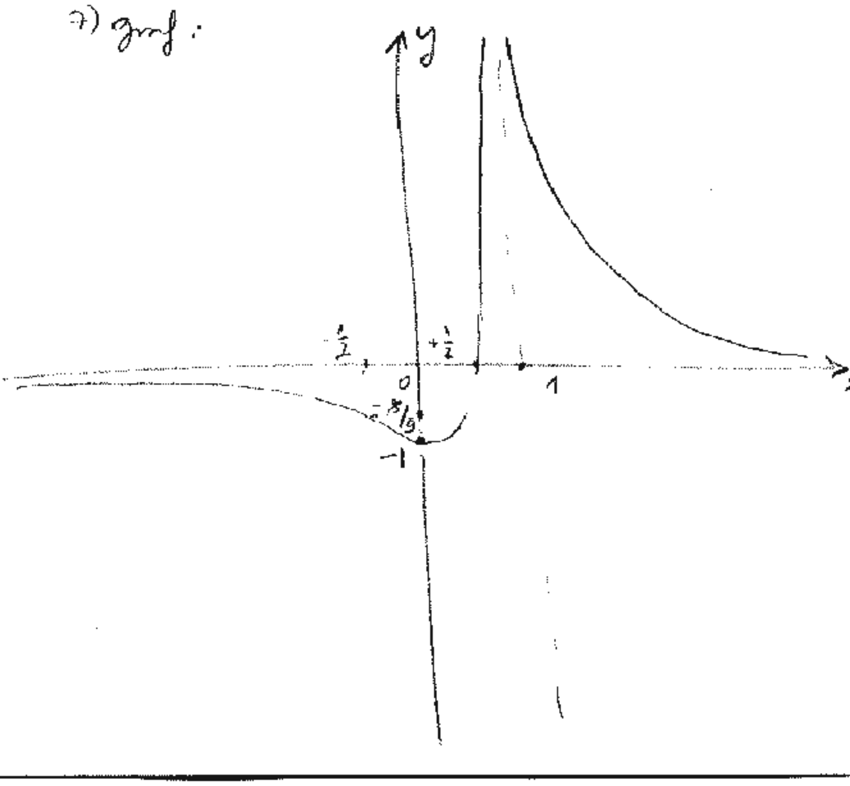
b) $x>0$
$x-1<0 \Rightarrow x<1$
$x \in \langle 0,1\rangle$

$\Rightarrow$ rast: $\boxed{x\in\langle 0,1\rangle}$

pad: $\boxed{x\in\langle -\infty,0\rangle \cup \langle 1,\infty\rangle}$

	$-\infty$	$\langle -\infty,0\rangle$	0	$\langle 0,1\rangle$	1	$\langle 1,\infty\rangle$	∞
f'	×	−	0	+	×	−	×
f	0	↘	Min	↗	×	↘	0

7) graf:



MATEMATIKA 1 – 21.6.2005 – RJEŠENJA

Zadatak. Odredite točku simetričnu ishodištu obzirom na pravac $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$.

Rješenje.

Povlačimo ravninu Π kroz $O(0,0,0)$ okomitu na zadani pravac

$\Rightarrow \quad \vec{n}=\vec{s}_{pravca}=\vec{i}+2\vec{j}+3\vec{k}$

$\Rightarrow \quad \Pi \ldots \ 1\cdot(x-0)+2\cdot(y-0)+3\cdot(z-0)=0$

$x+2y+3z=0$

$x=t+1,\ y=2t+1,\ z=3t+1,\ t\in\mathbb{R}$ – parametarska jednadžba pravca

pravac $\cap\ \Pi \ldots \quad t+1+2(2t+1)+3(3t+1)=0$

$t+4t+9t=-6 \Rightarrow t=-\frac{3}{7} \Rightarrow O'=\left(\frac{4}{7},\frac{1}{7},\frac{-2}{7}\right)$

$O''=?\qquad \vec{OO''}=2\vec{OO'},\quad O''=(x_0,y_0,z_0)$

$\Rightarrow \quad x_0\vec{i}+y_0\vec{j}+z_0\vec{k}=2\left(\frac{4}{7}\vec{i}+\frac{1}{7}\vec{j}-\frac{2}{7}\vec{k}\right) \Rightarrow \quad x_0=\frac{8}{7},\ y_0=\frac{2}{7},\ z_0=\frac{-4}{7}$

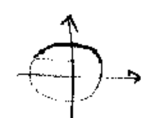
$\Rightarrow \boxed{O''=\left(\frac{8}{7},\frac{2}{7},-\frac{4}{7}\right)}$

Zadatak. Dokažite da u domeni funkcije $f(x)=\sqrt{\sin x-\frac{\sqrt{2}}{2}}+\sqrt{\cos x-\frac{\sqrt{2}}{2}}$ nema cjelobrojnih točaka.

Rješenje.

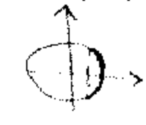
$\left.\begin{array}{l}\sin x-\frac{\sqrt{2}}{2}\geq 0\\ \cos x-\frac{\sqrt{2}}{2}\geq 0\end{array}\right\}$ zbog korijena $\Rightarrow$

$\sin x \geq \frac{\sqrt{2}}{2}$



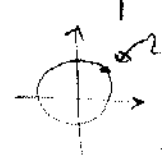
$x\in\bigcup_{k\in\mathbb{Z}}[2k\pi+\frac{\pi}{4},\ 2k\pi+\frac{3}{}$

$\cos x\geq\frac{\sqrt{2}}{2}$



$x\in\bigcup_{k\in\mathbb{Z}}[2k\pi-\frac{\pi}{4},\ 2k\pi+\frac{\pi}{4}]$

Vidimo da rješenje ovog sustava nejednadžbi predstavljaju sljedeće točke:



$\Rightarrow \quad \mathcal{D}(f)=\{2k\pi+\frac{\pi}{4}\mid k\in\mathbb{Z}\}$. Ovakve točke ne mogu biti cjelobrojne. Naime, ako pretpostavimo da za neki $k\in\mathbb{Z}$ vrijedi $2k\pi+\frac{\pi}{4}=m$ za neki $m\in\mathbb{Z}$ dobivamo $\pi\cdot\frac{8k+1}{4}=m \Rightarrow \pi=\frac{4m}{8k+1} \Rightarrow \pi\in\mathbb{Q}$ – kontradikcija.

$\Rightarrow$ U $\mathcal{D}(f)$ nema cjelobrojnih točaka.

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte $\lim_{x\to 1}\frac{(1-x)^2}{1-\sin\frac{\pi x}{2}}$.

Rješenje. $\lim\limits_{x\to 1}\dfrac{(1-x)^2}{1-\sin\frac{\pi x}{2}} = \begin{bmatrix} x-1=t \\ \Rightarrow t\to 0\end{bmatrix} = \lim\limits_{t\to 0}\dfrac{t^2}{1-\sin\frac{\pi(t+1)}{2}} =$

$$= \lim_{t\to 0}\frac{t^2}{1-\sin\left(\frac{\pi t}{2}+\frac{\pi}{2}\right)} = \lim_{t\to 0}\frac{t^2}{1-\cos\frac{\pi t}{2}} = \lim_{t\to 0}\frac{t^2}{2\sin^2\frac{\pi t}{4}} =$$

$\to$ po formuli $\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

$$= \frac{1}{2}\lim_{t\to 0}\frac{t^2\cdot\frac{\pi}{4}\cdot\frac{\pi}{4}\cdot\frac{4}{\pi}\cdot\frac{4}{\pi}}{\sin^2\frac{\pi t}{4}} =$$

$$= \frac{1}{2}\underbrace{\lim_{t\to 0}\frac{\left(\frac{\pi t}{4}\right)^2}{\sin^2\frac{\pi t}{4}}}_{1^2=1}\cdot\frac{16}{\pi^2} = \boxed{\frac{8}{\pi^2}}$$

Zadatak. Izračunajte približno $\sqrt{2.01^3+1}$.

Rješenje. $f(x)=\sqrt{x^3+1}$, $x_0=2$, $\Delta x = 0.01 \Rightarrow f'(x)=\frac{1}{2\sqrt{x^3+1}}\cdot 3x^2$

$$\sqrt{2.01^3+1} = f(2.01) \approx f(x_0)+f'(x_0)\cdot\Delta x = f(2)+f'(2)\cdot 0.01 =$$

$$= \sqrt{2^3+1}+\frac{1}{2\sqrt{2^3+1}}\cdot 3\cdot 2^2\cdot\frac{1}{100} = 3+\frac{1}{\cancel{2}\cdot\cancel{3}}\cdot\cancel{3}\cdot 2^{\cancel{2}}\cdot\frac{1}{100} = 3+\frac{1}{50} = \boxed{\frac{151}{50}}$$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{3x^4+1}{x^3}$.

Rješenje. $f(x) = 3x+\frac{1}{x^3} \Rightarrow \mathcal{D}(f)=\mathbb{R}\setminus\{0\}$; $N(f)=\emptyset$ (nazivnik je pozitivan!)

$f'(x) = 3-\frac{3}{x^4} = 0 \Rightarrow 3x^4 = 3 \;/:3 \Rightarrow x_{1,2}=\pm 1$ kandidati za lok. ekstreme

$f''(x) = \frac{12}{x^5} \Rightarrow f''(1) = 12>0 \Rightarrow (1,f(1)) = (1,4)$ lok. min.

$f''(-1) = -12<0 \Rightarrow (-1,f(-1)) = (-1,-4)$ lok. max.

asimptote: V.A. $x=0$

H.A. nema $\left(\lim\limits_{x\to\infty} f(x) = \infty\right)$

K.A. $y=kx+l$

$k = \lim\limits_{x\to\infty}\frac{f(x)}{x} = \lim\limits_{x\to\infty}\frac{3x^4+1}{x^4} = 3$

$l = \lim\limits_{x\to\infty}(f(x)-3x) = \lim\limits_{x\to\infty}\frac{\cancel{3x^4}+1-\cancel{3x^4}}{x^3} = 0$ $\Rightarrow y=3x$ kosa asimp.

rast-pad: rast $f'(x)>0 \Rightarrow 3-\frac{3}{x^4}>0 \Rightarrow 3>\frac{3}{x^4} \;/\cdot\frac{x^4}{3} \Rightarrow x^4>1$

$\Rightarrow x^4-1>0 \Rightarrow (x^2-1)\underbrace{(x^2+1)}_{>0\ \forall x\in\mathbb{R}}>0 \Rightarrow x^2>1 \Rightarrow |x|>1$

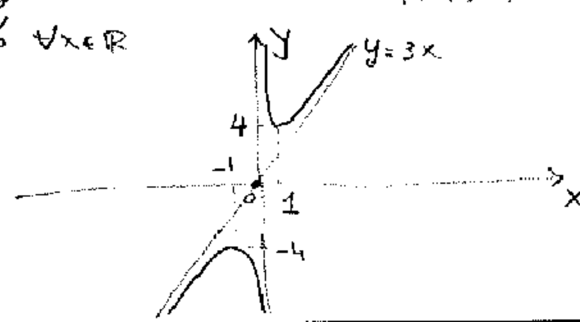
$\Rightarrow$ rast: $x\in\langle-\infty,-1\rangle\cup\langle 1,\infty\rangle$

pad: $x\in\langle -1,1\rangle\setminus\{0\}$

tok:

	$-\infty$	$\langle-\infty,-1\rangle$	-1	$\langle-1,0\rangle$	0	$\langle 0,1\rangle$	1	$\langle 1,\infty\rangle$	∞
f'	$\times$	$+$	0	$-$	$\times$	$-$	0	$+$	$\times$
f	$-\infty$	$\nearrow$	MAX	$\searrow$	$\times$	$\searrow$	MIN	$\nearrow$	$+\infty$

graf:



MATEMATIKA 1 – 5.7.2005. – RJEŠENJA

Zadatak. Izračunajte udaljenost presječne točke pravaca

$$p_1 \ldots \begin{cases} x-y+z=1 \\ 2x-y-z=1 \end{cases}, \quad p_2 \ldots \begin{cases} x+y-z=3 \\ -x+y+z=1 \end{cases}$$ od ishodišta.

Rješenje. Presječna točka T pravaca p_1 i p_2 je ona točka koja zadovoljava jednadžbe koje definiraju pravce p_1 i p_2 $\Rightarrow$ tražimo rješenje sustava

$$\begin{aligned} x-y+z&=1 \\ 2x-y-z&=1 \\ x+y-z&=3 \\ -x+y+z&=1 \end{aligned}$$

Radi se o sustavu od 4 jednadžbe s 3 nepoznanice, pa možemo rješavati bilo koje tri od njih, recimo

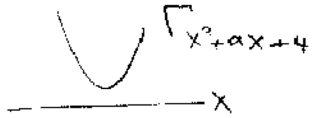
$$\begin{aligned} x-y+z&=1 \\ x+y-z&=3 \\ -x+y+z&=1. \end{aligned}$$

Zbrajanjem prve dvije izlazi $x=2$, druge dvije $y=2$, a prve i zadnje $z=1 \Rightarrow T=(2,2,1)$. Uvrštavanje u $2x-y-z=1$ potvrđuje da se p_1 i p_2 doista sijeku.

Tražimo $d=|\overline{TO}|=\sqrt{(2-0)^2+(2-0)^2+(1-0)^2}=\sqrt{9}=\boxed{3}$.

Zadatak. Nađite sve cijele brojeve a takve da domena funkcije $f(x)=\ln(x^2+ax+4)$ bude čitav skup realnih brojeva.

Rješenje. Jedini zahtjev na domenu funkcije f glasi $x^2+ax+4>0$, zbog logaritamske funkcije. Da $\mathcal{D}(f)$ bude čitav $\mathbb{R}$, nužno je i dovoljno da vrijedi



, što će biti točno ako i samo ako za diskriminantu ove kvadratne funkcije vrijedi

$$D=a^2-4\cdot1\cdot4=a^2-16<0$$ (nema realnih korijena, tj. presjeka s x-osi)

$$\Rightarrow \quad a^2<16 \Rightarrow \quad |a|<4 \Rightarrow \quad \boxed{a\in\{-3,-2,-1,0,1,2,3\}}$$

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte $\lim\limits_{x\to0}\dfrac{2\sin x-\sin 2x}{x^3}$.

Rješenje.

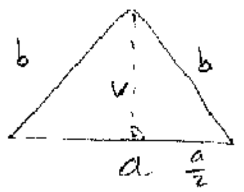
$$\lim_{x\to0}\frac{2\sin x-\sin 2x}{x^3}=\lim\frac{2\sin x-2\sin x\cdot\cos x}{x^3}=$$

$$=\lim_{x\to0}\frac{2\sin x(1-\cos x)}{x^3}=\lim_{x\to0}\frac{2\sin x\cdot 2\sin^2\frac{x}{2}}{x^3}=$$

$$=\lim_{x\to0}4\cdot\underbrace{\frac{\sin x}{x}}_{1}\cdot\underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot2}}_{1}\cdot\underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot2}}_{1}=4\cdot\frac{1}{2}\cdot\frac{1}{2}=\boxed{1}.$$

Zadatak. Među svim jednakokračnim trokutima zadanog opsega O odredite onaj maksimalne površine.

Rješenje.



$O = a+2b \Rightarrow a = O-2b$

$P = \frac{av}{2} \qquad v^2+\left(\frac{a}{2}\right)^2=b^2 \Rightarrow v^2=b^2-\frac{a^2}{4}$

$v=\sqrt{b^2-\frac{a^2}{4}}$

$P=\frac{1}{2}\cdot a\cdot v=\frac{1}{2}\cdot(O-2b)\cdot\sqrt{b^2-\frac{1}{4}(O-2b)^2}=$

$=\frac{1}{2}(O-2b)\cdot\sqrt{b^2-\frac{1}{4}(4b^2-4bO+O^2)}=$

$=\frac{1}{2}(O-2b)\cdot\sqrt{\cancel{b^2}-\cancel{b^2}+b\cdot O-\frac{O^2}{4}}=\frac{1}{4}(O-2b)\cdot\sqrt{4bO-O^2}$

$P(b)=\frac{1}{4}(O-2b)\cdot\sqrt{4bO-O^2} \Rightarrow P'(b)=\frac{1}{\cancel{4}_2}\cdot\cancel{2}\sqrt{4bO-O^2}+\frac{1}{\cancel{4}_2}(O-2b)\cdot\frac{1}{\cancel{2}\sqrt{4bO-O^2}}\cdot\cancel{4}^{1}O=0 \;\Big/\; 2\sqrt{4bO-O^2}$

$\Rightarrow \quad -4bO+O^2+(O-2b)\cdot O=0 \;/ : O\neq 0$

$-4b+O+O-2b=0 \Rightarrow b=\frac{O}{3}$

$\Rightarrow a=\frac{O}{3}$ $\Big\} \Rightarrow a=b$

$\Rightarrow$ radi se o jednakostraničnom trokutu

$P_{max}=\frac{1}{4}\left(O-\frac{2}{3}O\right)\sqrt{\left(\frac{O}{3}\right)^2-\frac{1}{4}\left(O-\frac{2}{3}O\right)^2}=\dots=\frac{O^2\sqrt{3}}{72}$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x)=\frac{2x-5}{(x-3)^2}$.

Rješenje. $D(f)=\mathbb{R}\setminus\{3\}$

$N(f)=\{5/2\}$

$-\frac{5}{9}$

asimptote: vertikalna: $x=3$

horizontalna: $y=\lim\limits_{x\to\infty}f(x)=\dots=0 \Rightarrow y=0$

kosa: nema

$f'(x)=\frac{2(x-3)^{\cancel{2}}-(2x-5)\cdot2\cancel{(x-3)}}{(x-3)^{\cancel{4}\,3}}=\frac{2x-6-4x+10}{(x-3)^3}=\frac{4-2x}{(x-3)^3}=0 \Rightarrow x=2$ kandidat za ekstrem

$f''(x)=\frac{-2(x-3)^{\cancel{3}}+(4-2x)\cdot3\cancel{(x-3)^2}}{(x-3)^{\cancel{6}\,4}}=\frac{-2x+6-6x+12}{(x-3)^4}=\frac{18-8x}{(x-3)^4}$

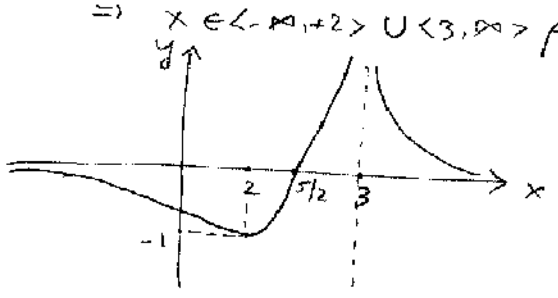
$f''(2)=\frac{18-16}{(2-3)^4}>0 \Rightarrow (2,f(2))=(2,-1)$ točka lok. minimuma

tok: $f'(x)>0 \Rightarrow \frac{4-2x}{(x-3)^3}=\frac{4-2x}{(x-3)\underbrace{(x-3)^2}_{\geq 0}}>0 \to \dots x\in\langle 2,3\rangle$ rast

$\Rightarrow x\in\langle-\infty,2\rangle\cup\langle 3,\infty\rangle$ pad

	$-\infty$	$\langle-\infty,2\rangle$	2	$\langle 2,3\rangle$	3	$\langle 3,\infty\rangle$	∞
f'	×	−	0	+	×	−	×
f	0	↘	MIN	↗	×	↘	0

graf:



MATEMATIKA 1 - 13.7.2005. - RJEŠENJA

Zadatak. Odredite jednadžbu ravnine koja sadrži točke $T_1(1,2,0)$ i $T_2(2,3,1)$, a okomita je na ravninu $2x+3y-4z=0$.

Rješenje. Treba nam vektor normale $\vec{n}$ ravnine čiju jednadžbu želimo odrediti. Možemo definirati $\vec{n} := \vec{v} \times \vec{n}_1$, gdje je $\vec{v} = \overrightarrow{T_1T_2} = \vec{i}+\vec{j}+\vec{k}$, a $\vec{n}_1 = 2\vec{i}+3\vec{j}-4\vec{k}$ (vektor normale zadane ravnine), pa imamo

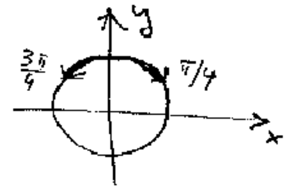
$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 3 & -4 \end{vmatrix} = -7\vec{i}+6\vec{j}+\vec{k}$. Kao točku u ravnini možemo uzeti $T_1(1,2,0)$, pa konačno imamo

$$-7(x-1)+6(y-2)+1(z-0)=0$$
$$-7x+7+6y-12+z=0 \quad /\cdot(-1)$$
$$\boxed{7x-6y-z=-5}$$

Zadatak. Odredite domenu funkcije $f(x) = \ln\sqrt{\sin(\arccos x) - \frac{\sqrt{2}}{2}}$

Rješenje.

- $\sin(\arccos x) - \frac{\sqrt{2}}{2} \geq 0$ (zbog $\sqrt{\ }$)
- $\sqrt{\sin(\arccos x) - \frac{\sqrt{2}}{2}} > 0$ (zbog $\ln$)



$\Rightarrow \quad \sin(\arccos x) - \frac{\sqrt{2}}{2} > 0 \Leftrightarrow \sin(\arccos x) > \frac{\sqrt{2}}{2}$

$\Rightarrow \quad \arccos x \in \bigcup_{k\in\mathbb{Z}} \langle 2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4} \rangle$

No, kako je $\mathcal{R}(\arccos) = [0,\pi]$, gledamo samo interval za $k=0$

$\Rightarrow \quad \arccos x \in \langle \frac{\pi}{4}, \frac{3\pi}{4} \rangle \Leftrightarrow \quad \frac{\pi}{4} < \arccos x < \frac{3\pi}{4} \quad / \cos$

$$\frac{\sqrt{2}}{2} > x > -\frac{\sqrt{2}}{2} \Rightarrow$$
$$\boxed{\mathcal{D}(f) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle}$$

Zadatak. Bez korištenja L'Hospitalovog pravila izračunajte $\lim\limits_{x\to 1} \frac{\cos(\frac{\pi x}{2})}{\sqrt{x}-1}$.

Rješenje. $\lim\limits_{x\to 1} \frac{\cos(\frac{\pi x}{2})}{\sqrt{x}-1} = \begin{bmatrix} t := x-1 \\ \Rightarrow t\to 0 \end{bmatrix} = \lim\limits_{t\to 0} \frac{\cos\frac{\pi}{2}(t+1)}{\sqrt{t+1}-1} =$

$$= \lim_{t\to 0} \frac{\cos(\frac{\pi}{2}t + \frac{\pi}{2})}{\sqrt{t+1}-1} = \lim_{t\to 0} \Big(\underbrace{-\frac{\sin\frac{\pi}{2}t}{\frac{\pi}{2}t}}_{1} \cdot \frac{\pi}{2}t \cdot \frac{1}{\sqrt{t+1}-1} \cdot \frac{\sqrt{t+1}+1}{\sqrt{t+1}+1} \Big) =$$

$$= \lim_{t\to 0} \Big(-\frac{\pi}{2}t \cdot \frac{\sqrt{t+1}+1}{t+\cancel{1}-\cancel{1}} \Big) = \lim_{t\to 0} \Big(-\frac{\pi}{2}\cancel{t} \cdot \frac{1}{\cancel{t}} \cdot \underbrace{(\sqrt{t+1}+1)}_{=2} \Big) = -\frac{\pi}{\cancel{2}} \cdot \cancel{2} = \boxed{-\pi}$$

Zadatak. Nađite sve lokalne ekstreme funkcije $f(x) = \ln\sqrt{x^5-5x}$.

Rješenje. Nužan uvjet za ekstrem: $f'(x)=0$

$$f'(x) = \frac{1}{\sqrt{x^5-5x}}\cdot\frac{1}{2\sqrt{x^5-5x}}\cdot(5x^4-5) = \frac{5(x^4-1)}{2(x^5-5x)} = 0$$

$$\Rightarrow \quad x^4-1=0 \Rightarrow (x^2-1)(\underbrace{x^2+1}_{\neq 0})=0 \to x^2=1 \Rightarrow x_{1,2}=\pm 1$$

No, $x_1=1$ ne pripada domeni funkcije, dok $x_1=-1$ pripada, pa je $x=-1$ jedini kandidat za lokalni ekstrem.

$$f''(x) = \frac{5\cdot 4x^3\cdot 2(x^5-5x) - 5(x^4-1)\cdot 2\cdot(5x^4-5)}{4(x^5-5x)^2}$$

$f''(-1) = \dots = -\frac{5}{2} < 0 \Rightarrow (-1, f(-1))$ lok. maksimum koji iznosi

$$f(-1) = \ln\sqrt{(-1)^5-5\cdot(-1)} = \underline{\ln 2}$$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \dfrac{2x^2+4x-1}{x^2+4x+5}$.

Rješenje. $N(f) = \{-1\pm\frac{\sqrt{6}}{2}\}$ $\quad 2x^2+4x-1=0 \Rightarrow x_{1,2} = -1\pm\frac{\sqrt{6}}{2}$

$D(f) = \mathbb{R}$ $\quad x^2+4x+5=0 \to$ nema realnih nultočaka

asimptote: vertikalna – nema

horizontalna $y = \lim\limits_{x\to\infty} f(x) = 2$

kosa – nema

$$f'(x) = \frac{(4x+4)(x^2+4x+5)-(2x^2+4x-1)\cdot(2x+4)}{(x^2+4x+5)^2} = \dots = \frac{4x^2+22x+24}{(x^2+4x+5)^2} \Rightarrow \begin{array}{l} x_1=-4 \\ x_2=-3/2 \end{array}$$

$$f''(x) = \frac{(8x+22)(x^2+4x+5)^{\cancel{2}} - (4x^2+22x+24)\cdot 2\cancel{(x^2+4x+5)}\cdot(2x+4)}{(x^2+4x+5)^{\cancel{4}\,3}} = \dots =$$

$$= \frac{-8x^3-66x^2-144x-82}{(x^2+4x+5)^3}$$

$f''(-4) = \dots < 0 \Rightarrow \boxed{\max\ f(-4)=3}$

$f''(-3/2) = \dots > 0 \Rightarrow \boxed{\min\ f(-3/2)=-2}$

rast-pad: rast $f'(x)>0 \to \dfrac{4x^2+22x+24}{\underbrace{(x^2+4x+5)^2}_{>0\ \forall x}} > 0 \Rightarrow 2(2x^2+11x+12)>0$

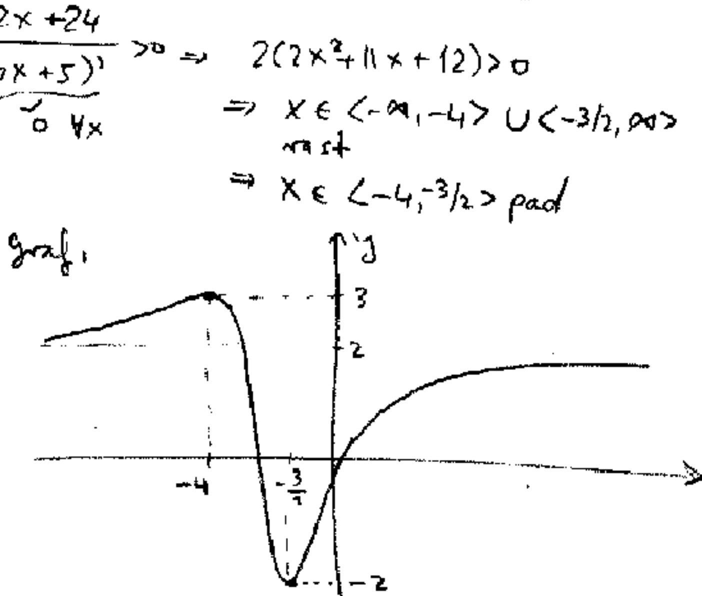
$\Rightarrow x \in \langle -\infty,-4\rangle \cup \langle -3/2,\infty\rangle$ rast

$\Rightarrow x \in \langle -4,-3/2\rangle$ pad

tok:

	$-\infty$	$\langle -\infty,-4\rangle$	-4	$\langle -4,-3/2\rangle$	$-3/2$	$\langle -3/2,\infty\rangle$	∞
f'		$+$	0	$-$	0	$+$	
f	2	$\nearrow$	MAX	$\searrow$	MIN	$\nearrow$	2

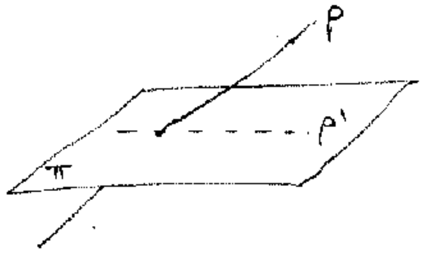
graf:



MATEMATIKA 1 – 26.9.2005. – RJEŠENJA

Zadatak. Odredite ortogonalnu projekciju pravca $p \ldots \begin{cases} x=1 \\ y=2 \end{cases}$ na ravninu $\Pi \ldots\ x+y+z=1$.

Rješenje



Želimo naći ravninu Π' koja sadrži p a okomita je na Π. Ako označimo sa $\vec{s}$ vektor smjera pravca p, a s $\vec{n}$ vektor normale ravnine Π, onda će vektor normale $\vec{n}'$ ravnine Π' biti $\vec{n}'=\vec{n}\times\vec{s}$.

$\vec{s}=?$ $\quad \begin{matrix} x=1 \Rightarrow & x-1=0\cdot t \\ y=2 \Rightarrow & y-2=0\cdot t \end{matrix} \Rightarrow \quad p \ldots \frac{x-1}{0}=\frac{y-2}{0}=\frac{z-0}{1} \Rightarrow \vec{s}=\vec{k}$

$\vec{n}'=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i}+\vec{j}$. Kao točku u ravnini Π' možemo uzeti bilo koju točku s pravca p, npr. $T=(1,2,0)$, pa Π' glasi $-1(x-1)+2(y-1)+0(z-0)=0$

$-x+2y=0$

$x-2y=0.$

Ort. projekciju p' dobivamo kao presjek Π i Π':

$\begin{matrix} x+y+z=1 \\ x-2y=0 \end{matrix} \Big|- \Rightarrow \quad 3y+z=1 \Rightarrow y=t,\ z=1-3t \\ x=2y=2t \Big\} \Rightarrow \boxed{p' \ldots \frac{x-0}{2}=\frac{y-0}{1}=\frac{z-1}{-3}}$

Zadatak. Nađite domenu funkcije $f(x)=\sqrt{4\sin^2x+4\sin x-3}$.

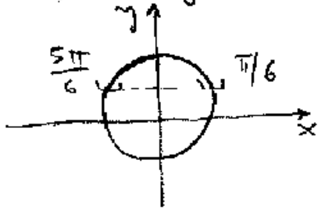
Rješenje. Jedini uvjet glasi $4\sin^2x+4\sin x-3\geq 0$, zbog korijena.

Supst. $t=\sin x \Rightarrow \quad 4t^2+4t-3\geq 0$

$D=16+4\cdot4\cdot3=64 \Rightarrow \quad t_{1,2}=\frac{-4\pm 8}{8}=\frac{-1\pm 2}{2} \Rightarrow t_1=\frac{-3}{2}$

$t_2=\frac{1}{2}$

$\Rightarrow \quad t\in\langle-\infty,-\frac{3}{2}]\cup[\frac{1}{2},\infty\rangle$. No, kako je $t=\sin x\in[-1,1]$, to je $\sin x\in[\frac{1}{2},1]$



$$\boxed{\mathcal{D}(f)=\bigcup_{k\in\mathbb{Z}}\left[\frac{\pi}{6}+2k\pi,\frac{5\pi}{6}+2k\pi\right]}$$

Zadatak. Bez upotrebe L'Hospitalovog pravila izračunajte limes

$$\lim_{x\to1}\frac{\tan\pi x}{\sqrt{x}-1}.$$

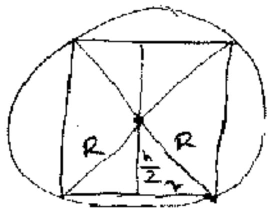
Rješenje. $\displaystyle \lim_{x\to1}\frac{\tan\pi x}{\sqrt{x}-1}=\begin{bmatrix} t=x-1 \\ t\to0 \end{bmatrix}=\lim_{t\to0}\frac{\tan(\pi t+\pi)}{\sqrt{t+1}-1}=$

$$= \lim_{t\to 0} \frac{\tan \pi t}{\sqrt{t+1}-1} \cdot \frac{\sqrt{t+1}+1}{\sqrt{t+1}+1} = \lim_{t\to 0} \frac{(\sqrt{t+1}+1)\cdot \tan \pi t}{t+\cancel{1}-\cancel{1}} =$$

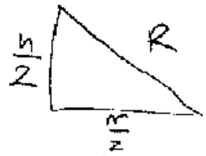
$$= \lim_{t\to 0} \frac{\overbrace{\sqrt{t+1}+1}^{\to 2}}{\underbrace{\cos \pi t}_{\to 1}} \cdot \underbrace{\frac{\sin \pi t}{\pi t}}_{\to 1} \cdot \pi = \boxed{2\pi}$$

Zadatak. U zadanu sferu polumjera R upišite valjak maksimalnog obujma.

Rješenje.



$$\left(\frac{h}{2}\right)^2 + r^2 = R^2$$
$$r^2 = R^2 - \frac{h^2}{4}$$



$$V = r^2 \pi \cdot h$$
$$V = \pi \cdot h \cdot \left(R^2 - \frac{h^2}{4}\right)$$
$$V' = \pi\left(R^2 - \frac{h^2}{4}\right) + \pi h \cdot \left(-\frac{h}{2}\right) = 0 \quad /:\pi$$
$$R^2 - \frac{h^2}{4} - \frac{h^2}{2} = 0 \Rightarrow 3h^2 = 4R^2$$
$$\boxed{h = \frac{2\sqrt{3}}{3}R}$$
$$r = 4R^2 - \frac{4R^2}{3} = \frac{8R^2}{3}$$
$$\boxed{r = \frac{2\sqrt{6}}{3}R}$$

$$\Rightarrow V_{max} = \frac{8R^2}{3} \cdot \pi \cdot \frac{2\sqrt{3}}{3}R$$
$$\boxed{V_{max} = \frac{16\sqrt{3}}{9}R^3\pi}$$

Zadatak. Ispitajte tok i nacrtajte graf funkcije $f(x) = \frac{3x^2-x}{x+1}$.

Rješenje. $\mathcal{D}(f) = \mathbb{R}\setminus\langle -1\rangle$

$\mathcal{N}(f) = ?$ $3x^2 - x = 0$, $x(3x-1) = 0 \Rightarrow x_1 = 0,\ x_2 = 1/3 \Rightarrow \mathcal{N}(f) = \{0, 1/3\}$

asimptote: vertikalna $x = -1$; horizontalna nema; kosa $y = kx + l$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3x^2-x}{x^2+x} = 3$$

$$l = \lim_{x\to\infty} (f(x) - 3x) = \lim_{x\to\infty} \frac{\cancel{3x^2} - x - \cancel{3x^2} - 3x}{x+1} = \lim_{x\to\infty} \frac{-4x}{x+1} = -4 \Rightarrow \underline{y = 3x-4}$$

$$f'(x) = \frac{(6x-1)(x+1) - (3x^2-x)}{(x+1)^2} = \frac{3x^2+6x-1}{(x+1)^2}$$

$$f'(x) = 0 \Rightarrow x_{1,2} = -1 \pm \frac{2}{3}\sqrt{3}$$

$$f''(x) = \frac{(6x+6)(x+1)^{\cancel{2}} - (3x^2+6x-1)\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}} =$$

$$= \frac{\cancel{6x^2} + \cancel{12x} + 6 - \cancel{6x^2} - \cancel{12x} + 2}{(x+1)^3} = \frac{8}{(x+1)^3} \Rightarrow f''\left(-1 \pm \frac{2}{3}\sqrt{3}\right) = \frac{8}{(\cancel{-1} \pm \frac{2}{3}\sqrt{3} + \cancel{1})^3} \gtrless 0 \ (\pm)$$

$$\Rightarrow \left(-1 \pm \tfrac{2}{3}\sqrt{3},\ f\left(-1 \pm \tfrac{2}{3}\sqrt{3}\right)\right) \quad \begin{matrix} \text{MAX (− predznak)} \\ \text{MIN (+ predznak)} \end{matrix}$$

tok:

	$-\infty$	$\langle -\infty, -1-\frac{2}{3}\sqrt{3}\rangle$	$-1-\frac{2}{3}\sqrt{3}$	$\langle -1-\frac{2}{3}\sqrt{3}, -1\rangle$	-1	$\langle -1, -1+\frac{2}{3}\sqrt{3}\rangle$	$-1+\frac{2}{3}\sqrt{3}$	$\langle -1+\frac{2}{3}\sqrt{3}, \infty\rangle$	∞
f'		+	0	−	X	−	0	+	
f		↗	MAX	↘	X	↘	MIN	↗	

rast-pad:

rast $f'(x) > 0$

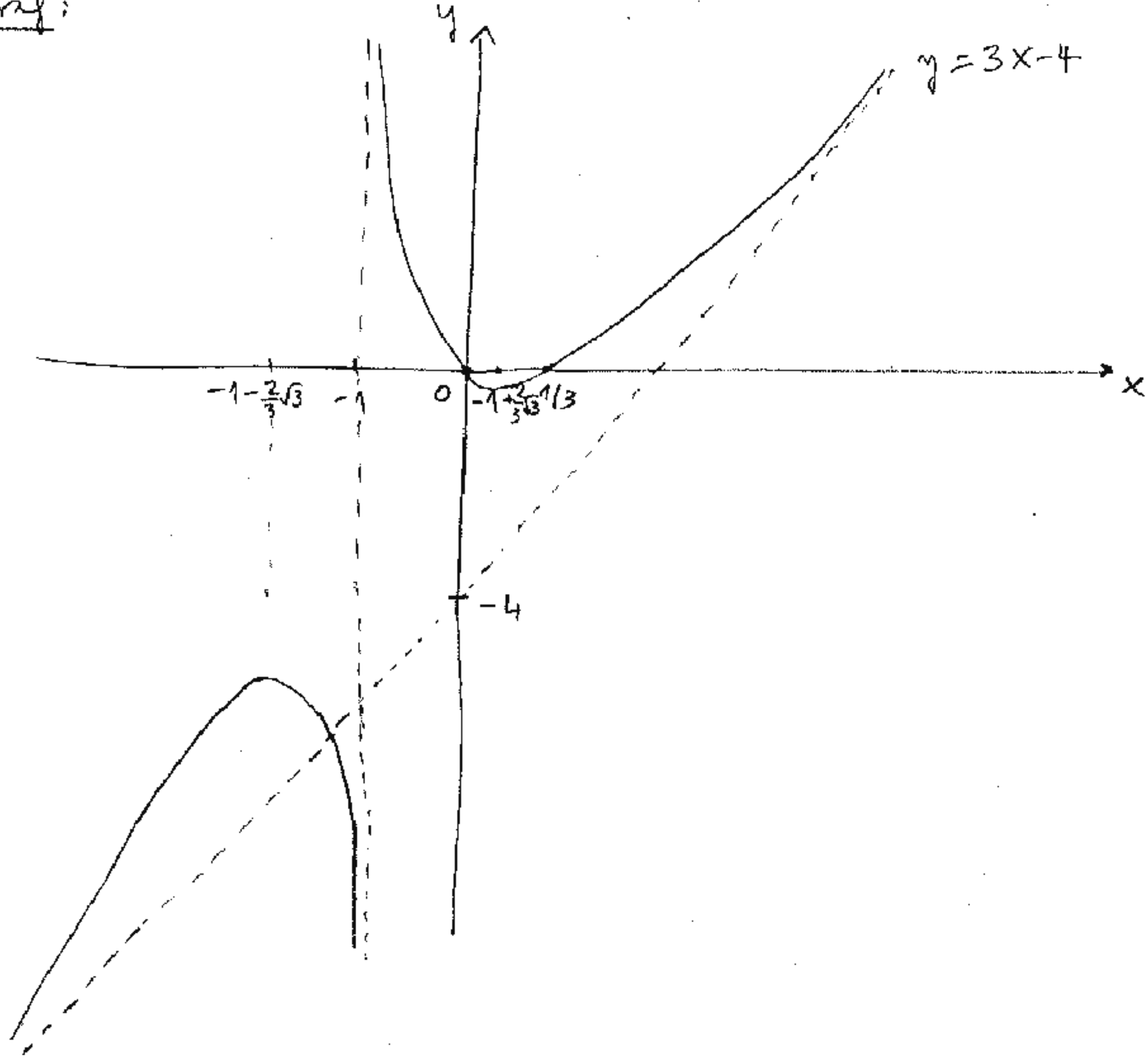
$$\Rightarrow \frac{3x^2+6x-1}{\underbrace{(x+1)^2}_{>0}} > 0$$

$$\Rightarrow 3x^2 + 6x - 1 > 0$$

$$\Rightarrow x \in \langle -\infty, -1-\tfrac{2}{3}\sqrt{3}\rangle \cup \langle -1+\tfrac{2}{3}\sqrt{3}, \infty\rangle$$

rast, ostalo pad

graf:



MATEMATIKA 1 – 3.10.2005. – RJEŠENJA

Zadatak. Kroz točku $(0,0,0)$ povucite pravac okomit pravcu

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1},$$ a paralelan ravnini $2x+3y+4z=5$.

Rješenje. Možemo vektor smjera $\vec{s}$ traženog pravca definirati kao vektorski produkt vektora smjera zadanog pravca i vektora normale zadane ravnine: $\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \vec{i} - 2\vec{j} + \vec{k}$.

Pravac glasi: $\frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$.

Zadatak. Nađite domenu funkcije $f(x) = \log \frac{5x^2-x^4-6}{x^4-4x^2+5}$.

Rješenje. $\frac{5x^2-x^4-6}{x^4-4x^2+5} > 0$

No, kod nazivnika je $x^4-4x^2+5 = x^4-4x^2+4+1 = (x^2-2)^2+1^2 > 0,\ \forall x$,

pa mora biti: $5x^2-x^4-6>0 \iff x^4-5x^2+6<0$

Supst. $x^2 = t$

$t^2-5t+6<0$

$t_{1,2} = \frac{5\pm\sqrt{25-24}}{2} = \frac{5\pm1}{2}$

$t_1 = 2,\ t_2 = 3 \Rightarrow t \in \langle 2,3\rangle$

$2 < x^2 < 3 \ /\sqrt{\ } \Rightarrow \sqrt{2} < |x| < \sqrt{3}$

$\rightarrow \boxed{\mathcal{D}(f) = \langle -\sqrt{3}, -\sqrt{2}\rangle \cup \langle \sqrt{2}, \sqrt{3}\rangle}$

Zadatak. Bez L'Hospitalovog pravila izračunajte limes $\lim_{x\to 2} \frac{\sin x - \sin 2}{x-2}$.

Rješenje. Supst. $t = x-2$

$$\lim_{x\to 2}\frac{\sin x-\sin 2}{x-2} = \lim_{t\to 0}\frac{\sin(t+2)-\sin 2}{t} = \lim_{t\to 0}\frac{\sin t\cos 2+\cos t\sin 2-\sin 2}{t} =$$

$$= \lim_{t\to 0}\left(\cos 2\cdot\frac{\sin t}{t} - \frac{\sin 2}{t}(1-\cos t)\right) =$$

$$= \lim_{t\to 0}\left(\cos 2\cdot\frac{\sin t}{t} - \frac{\sin 2}{t}\cdot 2\sin^2\frac{t}{2}\right) =$$

$$= \lim_{t\to 0}\left(\cos 2\cdot\underbrace{\frac{\sin t}{t}}_{1} - \sin 2\cdot\underbrace{\sin\frac{t}{2}}_{0}\cdot\underbrace{\frac{\sin\frac{t}{2}}{\frac{t}{2}}}_{1}\right) = \boxed{\cos 2}$$

Zadatak. Izračunajte približno $\sqrt[3]{8.01}$.

Rješenje. $f(x)=\sqrt[3]{x} \Rightarrow f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$

$x_0=8,\ \Delta x = 0.01$

$$\sqrt[3]{8.01}=f(x_0+\Delta x)\approx f(x_0)+\Delta x\cdot f'(x_0)=\sqrt[3]{8}+0.01\cdot\frac{1}{3\sqrt[3]{64}}=$$

$$=2+\frac{1}{100}\cdot\frac{1}{3\cdot 4}=\boxed{2+\frac{1}{1200}}.$$

Zadatak. Ispitajte tok i nacrtajte tok funkcije $f(x)=\frac{x^2-x}{(x+1)^2}$.

Rješenje. $\mathcal{D}(f)=\mathbb{R}\setminus\{-1\}$

$\mathcal{N}(f)=\{x \mid x^2-x=0\}=\{0,1\}$

asimptote: vertikalna: $x=-1$

horizontalna: $\lim_{x\to\infty} f(x)=1 \quad y=1$

kosa: nema

$$f'(x)=\frac{(2x-1)(x+1)^{\cancel{2}}-(x^2-x)\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}}}=\frac{2\cancel{x^2}+2x-x-1-2\cancel{x^2}+2x}{(x+1)^3}=$$

$$=\frac{3x-1}{(x+1)^3}=0 \qquad x=\frac{1}{3},$$

$$f''(x)=\frac{3(x+1)^{\cancel{3}}-(3x-1)\cdot 3(x+1)^{2}}{(x+1)^{\cancel{6}4}}=\frac{3x+3-9x+3}{(x+1)^4}=\frac{-6x+6}{(x+1)^4}$$

$$f''\left(\tfrac{1}{3}\right)=\frac{-\cancel{6}^{2}\cdot\frac{1}{\cancel{3}}+6}{(-\frac{1}{3}+1)^4}=\frac{4}{(2/3)^4}>0 \Rightarrow \left(\tfrac{1}{3}, f(\tfrac{1}{3})\right) \text{ lok. min.}$$

$$f\left(\tfrac{1}{3}\right)=\frac{\frac{1}{9}-\frac{1}{3}}{(\frac{1}{3}+1)^2}=\frac{\frac{-2}{\cancel{9}}}{\frac{16}{\cancel{9}}}=-\frac{1}{8} \Rightarrow \left(\tfrac{1}{3},-\tfrac{1}{8}\right)$$

rast pad: rast $f'(x)>0 \qquad \frac{3x-1}{(x+1)^3}=\frac{3x-1}{\underbrace{(x+1)^2}_{>0}\cdot(x+1)}>0$

$\frac{3x-1}{x+1}>0$ a) $3x-1>0,\ x+1>0 \Rightarrow x>1/3$

b) $3x-1<0,\ x+1<0 \Rightarrow x<-1$

$\Rightarrow x\in\langle-\infty,-1\rangle\cup\langle 1/3,\infty\rangle$ rast

$\Rightarrow x\in\langle -1, 1/3\rangle$ pad

tok:

	$-\infty$	$\langle-\infty,-1\rangle$	-1	$\langle-1,1/3\rangle$	$1/3$	$\langle 1/3,\infty\rangle$	∞
f'		$+$	$\times$	$-$	0	$+$	
f		$\nearrow$	$\times$	$\searrow$	$-\frac{1}{8}$	$\nearrow$	

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