MATEMATIKA 2

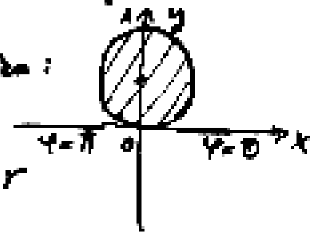
KOLOKVIJI 2003./04. - RJEŠENJA

3. kolokvij

MATEMATIKA 2 – 3 KOLOKVIJ - RJEŠENJA

1. Prelaskom na polarne koordinate riješite integral $\iint_S f(x,y)\,dx\,dy$, gdje je S područje omeđeno kružnicom $x^2+(y-1)^2=1$ i $f(x,y)=\sqrt{1-x^2-y^2}$

Rješenje. $x^2+(y-1)^2=1 \Rightarrow S=(0,1), r=1$ slika:



$x=r\cos\varphi,\ y=r\sin\varphi$

$x^2+y^2-2y=0 \Rightarrow r^2-2r\sin\varphi=0\ /:r$

$r=2\sin\varphi$

$\Rightarrow$ integral glasi: $\int_0^\pi d\varphi \int_0^{2\sin\varphi}\sqrt{1-r^2}\cdot r\,dr$

Najprije: $\int\sqrt{1-r^2}\cdot r\,dr = \begin{bmatrix}1-r^2=t^2\\ -2r\,dr=2t\,dt\end{bmatrix} = -\int t\cdot t\,dt = \frac{-t^3}{3} = -\frac{1}{3}(1-r^2)\sqrt{1-r^2}$,

pa je $\int_0^\pi d\varphi\int_0^{2\sin\varphi}\sqrt{1-r^2}\,r\,dr = -\int_0^\pi\left(\frac{1}{3}(1-r^2)\sqrt{1-r^2}\right)\Big|_0^{2\sin\varphi}d\varphi =$

$= -\frac{1}{3}\int_0^\pi\left((1-4\sin^2\varphi)\sqrt{1-4\sin^2\varphi}-(1-0)\sqrt{1-0}\right)d\varphi = -\frac{1}{3}\int_0^\pi(1-4\sin^2\varphi)\sqrt{1-4\sin^2\varphi}\,d\varphi+\frac{1}{3}\int_0^\pi d\varphi=\dots$

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2. Nađite volumen tijela omeđenog ravninama $x=0, y=0, z=0$ i $6x+3y+2z=6$.

Rješenje. Tražimo presjek ravnine $6x+3y+2z=6$ s x-y ravninom $\Rightarrow$ uvrštavamo $z=0$ i dobivamo $6x+3y=6$, tj. $2x+y=2$, tj. $y=2-2x$; presjek tog pravca s $y=0$ je $x=1$, pa integral glasi

$V=\int_0^1 dx\int_0^{2-2x}f(x,y)\,dy$, gdje je $f(x,y)=z=3-3x-\frac{3}{2}y$

$V=\int_0^1 dx\int_0^{2-2x}(3-3x-\frac{3}{2}y)\,dy = \int_0^1\left(3y-3xy-\frac{3}{2}\cdot\frac{y^2}{2}\right)\Big|_0^{2-2x}dx=$

$=\int_0^1\left(3(2-2x)-3x(2-2x)-\frac{3}{4}\cdot(2-2x)^2\right)dx=\int_0^1(3x^2-6x+3)\,dx=$

$=(x^3-3x^2+3x)\Big|_0^1=\boxed{1}$.

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3. Razvijte u red oko nule funkciju $f(x)=\frac{x}{x-3}$, nađite područje konvergencije tog reda i izračunajte $f^{(100)}(0)$.

Rješenje. $f(x)=\frac{x-3+3}{x-3}=1+\frac{3}{x-3}$

$\left.\begin{array}{l} f'(x)=3\cdot(-1)(x-3)^{-2} \\ f''(x)=3\cdot(-1)\cdot(-2)(x-3)^{-3}\end{array}\right\} \Rightarrow f^{(n)}(x)=3(-1)^n n!(x-3)^{-(n+1)}$

$$\Rightarrow f^{(n)}(0) = 3\cdot(-1)^n n!\cdot(0-3)^{-(n+1)} = 3\cdot(-1)^n n!\,(-1)^{-(n+1)}\, 3^{-(n+1)}$$

$$f^{(n)}(0) = -3\cdot n!\cdot 3^{-(n+1)}$$

$$f^{(n)}(0) = -n!\cdot 3^{-n}$$

$$\Rightarrow f^{(100)}(0) = -100!\cdot 3^{-100}$$

$$\boxed{f^{(100)}(0) = \frac{-100!}{3^{100}}}$$

Taylorov red glasi: $T(f(x)) = \sum_{n=0}^{\infty} \frac{-\cancel{n!}\cdot 3^{-n}}{\cancel{n!}}(x-0)^n$

$$T(f(x)) = \sum_{n=0}^{\infty} -3^{-n}\cdot x^n$$

$$\boxed{T(f(x)) = \sum_{n=0}^{\infty} -\left(\frac{x}{3}\right)^n}$$

Područje konvergencije:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{x}{3}\right|^n} = \left|\frac{x}{3}\right|$$

ako $\left|\frac{x}{3}\right| < 1$, tj. $-1 < \frac{x}{3} < 1 \Rightarrow \underline{-3 < x < 3}$ red konvergira

posebno provjeravamo za $\left|\frac{x}{3}\right| = 1$, tj. za $\underline{x=-3}$ i $\underline{x=3}$

$\underline{x=-3}$ red glasi $\sum_{n=0}^{\infty} -(-1)^n = \sum_{n=0}^{\infty}(-1)^{n+1}$ – divergira

$\underline{x=3}$ red glasi $\sum_{n=0}^{\infty} -1^n = \sum_{n=0}^{\infty}(-1)$ – divergira

$\Rightarrow$ područje konvergencije je $\boxed{\langle -3,3\rangle}$.

__Zadatak__. Riješite Cauchyjev problem $xy' + y - e^x = 0$, $y(1) = 0$

__Rješenje__.

$$xy' + y = e^x \quad /:x$$

$$y' + \frac{1}{x}\cdot y = \frac{1}{x}e^x \quad (*)$$

Rješavamo najprije homogenu jednadžbu $y' + \frac{1}{x}y = 0$, $y' = \frac{dy}{dx}$

$$\frac{dy}{dx} + \frac{1}{x}y = 0 \Rightarrow \frac{1}{y}dy = -\frac{1}{x}dx \quad /\int$$

$$\ln|y| = -\ln|x| + C_1$$

$$|y| = e^{C_1}\cdot e^{-\ln|x|}$$

$$|y| = C|x|^{-1}$$

$$y = \frac{C}{x}$$

Uzmimo sada da je $C = C(x)$ funkcija, tj. $y = \frac{C(x)}{x}$:

Uvrstimo u $(*)$ uz $y' = \frac{C'(x)\cdot x - C(x)}{x^2}$

$$\frac{C'(x)\cdot x - C(x)}{x^2} + \frac{1}{x}\cdot\frac{C(x)}{x} = \frac{1}{x}\cdot e^x \quad /\cdot x^2$$

$\Rightarrow C'(x)x = xe^x \Rightarrow C'(x) = e^x \Rightarrow C(x) = e^x + C$, pa je konačno rješenje(x) dano s $y = \frac{e^x}{x} + \frac{c}{x}$

Kako je $y(1) = 0 = e + c$, to je $c = -e$, pa je $\boxed{y = \frac{e^x - e}{x}}$

Zadatak. Riješite diferencijalnu jednadžbu $y'' - 2y' = \cos x$.

Rješenje. Rješenje glasi $y = y_H + y_P$, gdje je y_H rješenje homogene jednadžbe $y'' - 2y' = 0$, a y_P partikularno rješenje.

Najprije rješavamo homogenu jednadžbu $y'' - 2y' = 0$ – karakteristična jednadžba glasi $\lambda^2 - 2\lambda = 0$, a rješenja su $\lambda_1 = 0, \lambda_2 = 2$, pa je

$y_H = C_1 \cdot e^{0 \cdot x} + C_2 \cdot e^{2 \cdot x} \Rightarrow y_H = C_1 + C_2 e^{2x}$, C_1 i C_2 realne konstante.

Partikularno rješenje tražimo u obliku $y_P = A \sin x + B \cos x$

$\Rightarrow y_P' = A \cos x - B \sin x$

$y_P'' = -A \sin x - B \cos x$

Uvrštavamo y_P' i y_P'' u diferencijalnu jednadžbu da odredimo A i B:

$-A \sin x - B \cos x - 2(A \cos x - B \sin x) = \cos x$

$(2B - A) \sin x + (-2A - B - 1) \cos x = 0$

$$\Rightarrow \begin{aligned} -A + 2B &= 0 \\ -2A - B &= 1 \quad / \cdot 2 \quad / + \end{aligned}$$

$-5A = 2 \Rightarrow \boxed{A = -\frac{2}{5}} \Rightarrow \boxed{B = -\frac{1}{5}} \Rightarrow y_P = -\frac{1}{5}(2 \sin x + \cos x)$, pa

je značno rješenje dano s $\boxed{y = C_1 + C_2 \cdot e^{2x} - \frac{1}{5}(2 \sin x + \cos x), \quad C_1, C_2 = \text{konst.}}$