MATEMATIKA 2

KOLOKVIJI 2004./05. - RJEŠENJA

1. kolokvij
2. kolokvij
3. kolokvij

Zadatak. Izračunajte integral $\int \frac{dx}{x^3+x^2+2x-4}$.

Napomena: ako postoje cjelobrojne nultočke polinoma mogu se lako pogoditi ako pogledamo djelitelje slobodnog člana.

Rješenje. Djelitelji broja -4 su $\pm 1, \pm 2$ i lako vidimo da je $x=1$ nultočka polinoma u nazivniku $\Rightarrow$ $x^3+x^2+2x-4=(x-1)\cdot f(x)$

$$f(x)=\frac{x^3+x^2+2x-4}{x-1} \Rightarrow (x^3+x^2+2x-4):(x-1)=x^2+2x+4$$

$$\begin{array}{l} -(x^3-x^2) \\ \quad 2x^2+2x-4 \\ \quad -(2x^2-2x) \\ \qquad 4x-4 \\ \qquad -(4x-4) \\ \qquad\quad 0 \end{array}$$

$$\Rightarrow x^3+x^2+2x-4=(x-1)\cdot\underbrace{(x^2+2x+4)}$$

$D=4-16<0 \Rightarrow$ x^2+2x+4 se ne može faktorizirati u produkt 2 linearna realna faktora

Tražimo rastav $\frac{1}{x^3+x^2+2x-4}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+4}=\frac{Ax^2+2Ax+4A+Bx^2-Bx+Cx-C}{(x-1)(x^2+2x+4)}$

$$\Rightarrow \begin{array}{l} A+B=0 \Rightarrow B=-A \\ 2A-B+C=0 \\ 4A-C=1 \end{array} \Rightarrow \begin{array}{l} 3A+C=0 \\ 4A-C=1 \end{array} \Rightarrow 7A=1 \Rightarrow A=\frac{1}{7},\ B=-\frac{1}{7},\ C=\frac{-3}{7}$$

$$\Rightarrow \int\frac{dx}{x^3+x^2+2x-4}=\frac{1}{7}\int\frac{dx}{x-1}-\frac{1}{7}\int\frac{x+3}{x^2+2x+4}dx=\frac{1}{7}\ln|x-1|-\frac{1}{7}\cdot\frac{1}{2}\int\frac{2x+2+4}{x^2+2x+4}dx=$$

$$=\frac{1}{7}\ln|x-1|-\frac{1}{14}\underbrace{\int\frac{(2x+2)dx}{x^2+2x-4}}-\frac{2}{7}\int\frac{dx}{\underbrace{x^2+2x+4}}=$$

Uz supst $x^2+2x-4=t$, $(2x+2)dx=dt$ $\Rightarrow$ imamo $\int\frac{dt}{t}=\ln|t|=\ln|x^2+2x-4|$

$=(x+1)^2+4=(x+1)^2+\sqrt{3}^2$ $\Rightarrow$ rješavamo po formuli $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\,\mathrm{arctg}\frac{x}{a}$

$$=\frac{1}{7}\ln|x-1|-\frac{1}{14}\ln(x^2+2x-4)-\frac{2}{7}\cdot\frac{1}{\sqrt{3}}\,\mathrm{arctg}\frac{x+1}{\sqrt{3}}+C$$

Zadatak. Izračunajte određeni integral $\int_0^{1/2} x\,\mathrm{arctg}(x^2)\,dx$.

Rješenje. Računamo najprije pripadni neodređeni integral:

$$\int x\,\mathrm{arctg}(x^2)dx=\left[\begin{array}{l} x^2=t \\ 2x\,dx=dt \end{array}\right]=\frac{1}{2}\int\mathrm{arctg}\,t\,dt=\left[\begin{array}{ll} \text{parc. int.} & \\ u=\mathrm{arctg}\,t & dv=dt \\ \Downarrow & \Downarrow \\ du=\frac{1}{1+t^2}dt & v=t \end{array}\right]=$$

$$=\frac{1}{2}\left(t\,\mathrm{arctg}\,t-\underbrace{\int\frac{t}{1+t^2}dt}\right)=$$

uz supst. $1+t^2=s$ je $2t\,dt=ds$, pa imamo $\frac{1}{2}\int\frac{ds}{s}=\frac{1}{2}\ln|s|=\frac{1}{2}\ln|1+t^2|$

$$=\frac{1}{2}t\,\mathrm{arctg}\,t-\frac{1}{4}\ln|1+t^2|+C=\frac{1}{2}x^2\mathrm{arctg}(x^2)-\frac{1}{4}\ln|1+x^4|+C$$

Sada je $\int_0^{1/2} x\,\mathrm{arctg}(x^2)\,dx = \frac{1}{2}\left(x^2\mathrm{arctg}(x^2)\right)\Big|_0^{1/2} - \frac{1}{4}\left(\ln|1+x^4|\right)\Big|_0^{1/2} =$

$$= \frac{1}{2}\cdot\frac{1}{4}\,\mathrm{arctg}\,\frac{1}{4} - \frac{1}{4}\left(\ln\left(1+\frac{1}{16}\right) - \underbrace{\ln 1}_{=0}\right) = \underline{\frac{1}{8}\,\mathrm{arctg}\,\frac{1}{4} - \frac{1}{4}\ln\frac{17}{16}}$$

Zadatak. Izračunajte određeni integral $\int_{\ln\frac{1}{2}}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

Rješenje. U $x=0$ nazivnik $\sqrt{1-e^{2x}} = \sqrt{1-e^0} = 0$, pa je to nepravi integral. Imamo:

$$\int_{\ln\frac{1}{2}}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \lim_{\varepsilon\to 0_-}\left(\int_{\ln\frac{1}{2}}^{\varepsilon} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx\right) = (*)$$

Riješimo pripadni neodređeni integral:

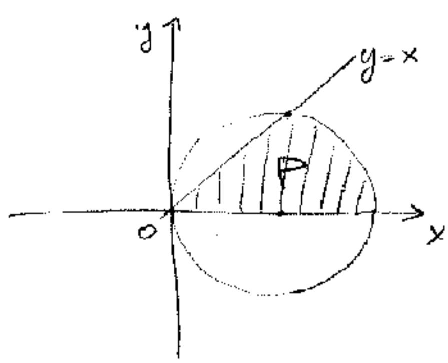
$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \begin{bmatrix} e^x = t \\ e^x dx = dt \end{bmatrix} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin e^x + C$$

$$\Rightarrow \quad (*) = \lim_{\varepsilon\to 0_-}\left((\arcsin e^x)\Big|_{\ln\frac{1}{2}}^{\varepsilon}\right) = \lim_{\varepsilon\to 0_-}\left(\arcsin e^{\varepsilon} - \arcsin \underbrace{e^{\ln\frac{1}{2}}}_{=\frac{1}{2}}\right) =$$

$$= \arcsin e^0 - \arcsin\frac{1}{2} = \arcsin 1 - \arcsin\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

Zadatak. Koristeći polarne koordinate izračunajte površinu omeđenu krivuljama $(x-1)^2+y^2=1$, $y=x$, $y=0$.

Rješenje. $(x-1)^2+y^2=1$ je kružnica sa središtem u $(0,1)$ i $r=1$, pa slika izgleda ovako.



Napišimo jednadžbu kružnice u polarnim koordinatama:

$x = r\cos\varphi$, $y = r\sin\varphi$

$$\Rightarrow \underbrace{r^2\cos^2\varphi - 2r\cos\varphi + r^2\sin^2\varphi}_{} = 0 \;/:r$$

$$\Rightarrow \underline{r = 2\cos\varphi}$$

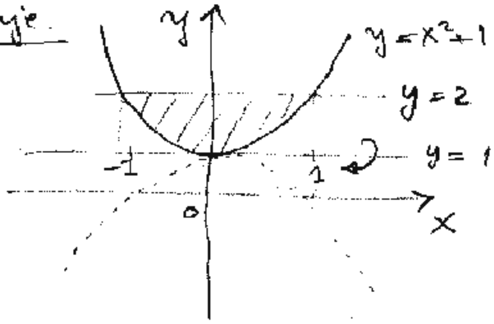
Očito su granice integracije $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{4}$ ($\leftrightarrow y=x$ pravac!), pa je

$$P = \frac{1}{2}\int_0^{\pi/4}(2\cos\varphi)^2\,d\varphi = \frac{1}{2}\int_0^{\pi/4} 4\cos^2\varphi\,d\varphi = \frac{1}{2}\int_0^{\pi/4} \cancel{4}_2\cdot\frac{1+\cos 2\varphi}{\cancel{2}}\,d\varphi =$$

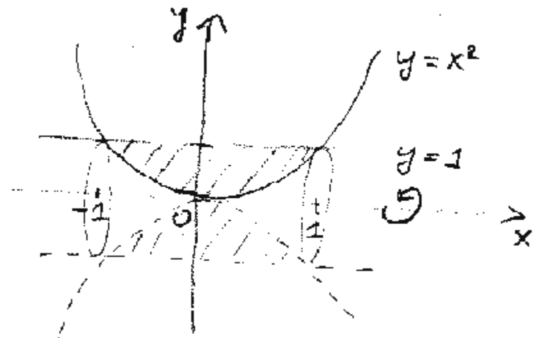
$$= \int_0^{\pi/4}(1+\cos 2\varphi)\,d\varphi = \left(\varphi + \frac{\sin 2\varphi}{2}\right)\Big|_0^{\pi/4} = \frac{\pi}{4} + \frac{\sin\pi/2}{2} - 0 = \boxed{\frac{\pi}{4} + \frac{1}{2}}$$

Zadatak. Izračunajte volumen tijela koje nastaje rotacijom područja omeđenog krivuljama $y = x^2+1$ i $y = 2$ oko pravca $y = 1$.

Rješenje.



Vidimo da je ovaj problem analogan problemu rotacije područja omeđenog krivuljama $y=x^2$ i $y=1$ oko pravca $y=0$, tj. x-osi:



Sada je traženi volumen: $V=V_1-V_2$, gdje je V_1 volumen dobiven rotacijom dijela pravca $y=1$ za $x\in[-1,1]$ oko x-osi, a V_2 volumen dobiven rotacijom dijela krivulje $y=x^2$ za $x\in[-1,1]$ oko x-osi.

$$\Rightarrow \left.\begin{array}{l} V_1=\pi\int_{-1}^{1}1^2dx=\pi\cdot(x)\Big|_{-1}^{1}=2\pi \\ V_2=\pi\int_{-1}^{1}(x^2)^2dx=\pi\cdot\left(\frac{x^5}{5}\right)\Big|_{-1}^{1}=\frac{2}{5}\pi \end{array}\right\} \Rightarrow V=2\pi-\frac{2}{5}\pi \Rightarrow \boxed{V=\frac{8}{5}\pi}$$

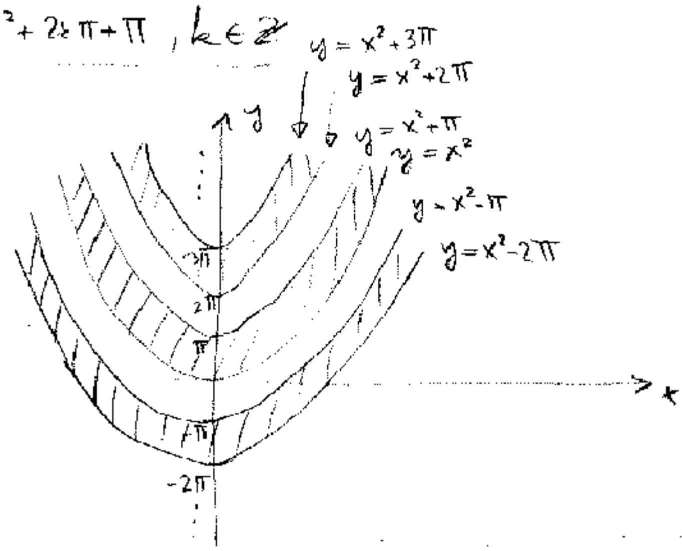
Zadatak. Skicirajte u koordinatnoj ravnini domenu funkcije $f(x,y)=(g\circ h)(x,y)$, ako je $g(t)=\sqrt{t}$, a $h(x,y)=\sin(y-x^2)$.

Rješenje. $f(x,y)=g(h(x,y))=g(\sin(y-x^2))=\sqrt{\sin(y-x^2)}$, pa imamo uvjet $\sin(y-x^2)\geq 0$, zbog korijena $\Rightarrow \quad y-x^2\in \bigcup\limits_{k\in\mathbb{Z}}[2k\pi,\, 2k\pi+\pi]$, tj.

$$2k\pi \leq y-x^2 \leq 2k\pi+\pi \quad \Big/ +x^2$$

$$\underline{x^2+2k\pi \leq y \leq x^2+2k\pi+\pi}, \; k\in\mathbb{Z}$$

Skica domene:



Zadatak. Na plohi $x^2+2y^2+3z^2=1$ nađite točke u kojima je tangencijalna ravnina paralelna s ravninom $4x+2y+6z=4$.

Rješenje. Tangencijalna ravnina na danu plohu u točki (x_0,y_0,z_0) te plohe glasi:

$$F_x(x_0,y_0,z_0)\cdot(x-x_0)+F_y(x_0,y_0,z_0)\cdot(y-y_0)+F_z(x_0,y_0,z_0)\cdot(z-z_0)=0, \text{ gdje}$$

je $F_x(x,y,z)=2x \Rightarrow F_x(x_0,y_0,z_0)=2x_0$

$F_y(x,y,z)=4y \Rightarrow F_y(x_0,y_0,z_0)=4y_0$

$F_z(x,y,z)=6z \Rightarrow F_z(x_0,y_0,z_0)=6z_0$,

pa je vektor normale te ravnine jednak $\vec{n}=2x_0\vec{i}+4y_0\vec{j}+6z_0\vec{k}$. Kako je tangencijalna ravnina paralelna s $4x+2y+6z=4$ kojoj vektor normale glasi $\vec{n}_1=4\vec{i}+2\vec{j}+6\vec{k}$), to su $\vec{n}$ i $\vec{n}_1$ kolinearni, tj. vrijedi:

$$\frac{2x_0}{4}=\frac{4y_0}{2}=\frac{6z_0}{6}, \text{ tj. } \quad \frac{x_0}{2}=2y_0=z_0 \quad /\cdot 2$$

$$x_0=4y_0=2z_0 \Rightarrow \quad x_0=4y_0$$

$$z_0=2y_0, \text{ što uvrštavanjem}$$

u jednadžbu plohe daje: $\quad (4y_0)^2+2y_0^2+3\cdot(2y_0)^2=1$

$$30y_0^2=1 \Rightarrow \quad y_0=\frac{\pm\sqrt{30}}{30} \Rightarrow x_0=\frac{\pm 2\sqrt{30}}{15},\; z_0=\frac{\pm\sqrt{30}}{15},$$

pa su tražene točke jednake $\boxed{T_{1,2}=\left(\pm\frac{2\sqrt{30}}{15},\, \pm\frac{\sqrt{30}}{30},\, \pm\frac{\sqrt{30}}{15}\right)}$.

Zadatak. Izračunajte približno $\sqrt{5.9+\sqrt[3]{1-3.1^2}}$

Rješenje. Definirajmo $f(x,y)=\sqrt{x+\sqrt[3]{1-y^2}}$. Tražimo $f(5.9,\,3.1)=f(6-0.1,\,3+0.1)$, tj.

$f(x_0+\Delta x,\,y_0+\Delta y)$, gdje je $x_0=6$, $\Delta x=-0.1$, $y_0=3$, $\Delta y=0.1$. Formula glasi

$$f(x_0+\Delta x,\,y_0+\Delta y)\approx f(x_0,y_0)+f_x(x_0,y_0)\cdot\Delta x+f_y(x_0,y_0)\cdot\Delta y.$$

$$f_x(x,y)=\frac{1}{2\sqrt{x+\sqrt[3]{1-y^2}}}\cdot 1 \Rightarrow f_x(6,3)=\frac{1}{2\sqrt{6+\sqrt[3]{1-3^2}}}=\frac{1}{2\sqrt{6-2}}=\frac{1}{2\sqrt4}=\frac14$$

$$f_y(x,y)=\frac{1}{2\sqrt{x+\sqrt[3]{1-y^2}}}\cdot\frac13(1-y^2)^{-2/3}\cdot(-2y)\Rightarrow f_y(6,3)=\frac{1}{\cancel{4}_{2}}\cdot\frac{\cancel{1}}{\cancel{3}}\cdot\frac{1}{\sqrt[3]{(1-9)^2}}\cdot(-\cancel{2}\cancel{3})=$$

$$=-\frac12\cdot\frac{1}{\sqrt[3]{64}}=-\frac12\cdot\frac14=-\frac18.$$

$$\Rightarrow\quad \sqrt{5.9+\sqrt[3]{1-3.1^2}}\approx f(6,3)+f_x(6,3)\cdot\frac{-1}{10}+f_y(6,3)\cdot\frac{1}{10}=$$

$$=\sqrt{6+\sqrt[3]{1-9}}+\frac14\cdot\frac{-1}{10}-\frac18\cdot\frac{1}{10}=$$

$$=2-\frac{1}{40}-\frac{1}{80}=2-\frac{3}{80}=\boxed{\frac{157}{80}}.$$

Zadatak. Odredite lokalne ekstreme funkcije $z=z(x,y)$ zadane implicitno s $x^2+2x+y^2+z^2-z=0$.

Rješenje. Definiramo $F(x,y,z)=x^2+2x+y^2+z^2-z$

Sada je $$z_x=-\frac{F_x}{F_z}=\frac{-2x-2}{2z-1}$$

$$z_y=-\frac{F_y}{F_z}=\frac{-2y}{2z-1}.$$ Nužan uvjet za ekstrem glasi

$$z_x(x_0,y_0)=z_y(x_0,y_0)=0,$$

pa imamo $\begin{cases}-2x-2=0\\-2y=0\end{cases}\Rightarrow\quad \begin{matrix}x=-1\\y=0\end{matrix}\Rightarrow$ uvrstimo u jednadžbu $x^2+2x+y^2+z^2-z=0$

i dobivamo $(-1)^2-2+z^2-z=0$

$z^2-z-1=0$

$z_{1,2}=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt5}{2}$

$\Rightarrow$ došli smo točke $T_1\left((-1,0),\,\frac{1-\sqrt5}{2}\right)$, $T_2\left((-1,0),\,\frac{1+\sqrt5}{2}\right)$.

$$z_{xx}(x,y)=\frac{-2\cdot(2z-1)-(-2x-2)\cdot2z_x}{(2z-1)^2}$$

$$z_{yx}(x,y)=z_{xy}(x,y)=\frac{0\cdot(2z-1)-(-2x-2)\cdot2z_y}{(2z-1)^2}=\frac{(2x+2)\cdot2z_y}{(2z-1)^2}$$

$$z_{yy}(x,y)=\frac{-2(2z-1)+2y\cdot2z_y}{(2z-1)^2}$$

$$\Rightarrow\quad H=\begin{bmatrix}\frac{-2(2z-1)-(-2x-2)2z_x}{(2z-1)^2} & \frac{(2x+2)\cdot2z_y}{(2z-1)^2}\\[2mm] \frac{(2x+2)2z_y}{(2z-1)^2} & \frac{-2(2z-1)+2y\cdot2z_y}{(2z-1)^2}\end{bmatrix}$$

$\boxed{T_1}$: $H\big|_{T_1} = \begin{bmatrix} \frac{-2\cdot(\cancel{1}-\sqrt5-\cancel{1})}{(\cancel{1}-\sqrt5-\cancel{1})^2} & 0 \\ 0 & \frac{-2\cdot(\cancel{1}-\sqrt5-\cancel{1})}{(\cancel{1}-\sqrt5-\cancel{1})^2} \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt5}{5} & 0 \\ 0 & \frac{2\sqrt5}{5} \end{bmatrix}$

$z_x\big|_{T_1} = z_y\big|_{T_1} = 0$!

$\Rightarrow \quad \Delta = \frac{4}{5} > 0 \Rightarrow$ postiže se lok. ekstrem i to (jer je $A = \frac{2\sqrt5}{5} > 0$) lok. minimum i iznosi $z_{min} = \frac{1-\sqrt5}{2}$

$\boxed{T_2}$: $H\big|_{T_2} = \begin{bmatrix} \frac{-2(\cancel{1}+\sqrt5-\cancel{1})}{(\cancel{1}+\sqrt5-\cancel{1})^2} & 0 \\ 0 & \frac{-2(\cancel{1}+\sqrt5-\cancel{1})}{(\cancel{1}+\sqrt5-\cancel{1})^2} \end{bmatrix} = \begin{bmatrix} \frac{-2\sqrt5}{5} & 0 \\ 0 & \frac{-2\sqrt5}{5} \end{bmatrix}$

$z_x\big|_{T_2} = z_y\big|_{T_2} = 0$!

$\Rightarrow \quad \Delta = \frac{4}{5} > 0 \Rightarrow$ postiže se lok. ekstrem i to (jer je $A = -\frac{2\sqrt5}{5} < 0$) lok. maksimum i iznosi $z_{max} = \frac{1+\sqrt5}{2}$

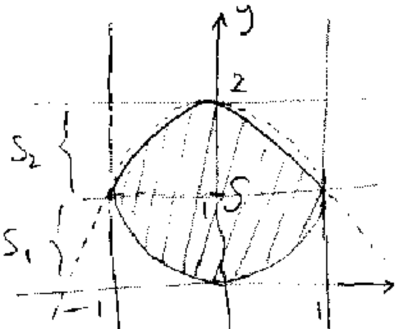
Zadatak. Promijenite redoslijed integracije i izračunajte integral $\int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{2-x^2} x\,dy$.

Rješenje. Crtamo područje integracije S: $x=-1$, $y = 1-\sqrt{1-x^2}$ (*), $x = 1$, $y = 2-x^2$

(*) $y = 1-\sqrt{1-x^2} \Rightarrow \sqrt{1-x^2} = 1-y \;/()^2$

$1-x^2 = (y-1)^2 \Rightarrow (y-1)^2+x^2 = 1^2 \Rightarrow$ svodi se na jednadžbu kružnice sa središtem u $(0,1)$ i radijusom $r=1$

$\Rightarrow$ (*) predstavlja donju polukružnicu, pa S izgleda ovako:



Promatramo sada S s y-osi – očito ga moramo rastaviti na 2 lika:

$\boxed{S_1}$ $y=0$, $y=+1$ | desna polukružnica: $(y-1)^2+x^2=1$; $y^2-2y+\cancel{1}+x^2=\cancel{1}$; $x^2 = 2y-y^2$; $x = \sqrt{2y-y^2}$ (gornja granica)

lijeva polukružnica: $x = -\sqrt{2y-y^2}$ (donja granica)

$\boxed{S_2}$ $y=1$, $y=2$ | desni krak parabole: $y = 2-x^2$; $x^2 = 2-y$; $x = \sqrt{2-y}$ (gornja granica)

lijevi krak parabole: $x = -\sqrt{2-y}$ (donja granica)

Konačno imamo:

$$\int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{2-x^2} x\,dy =$$

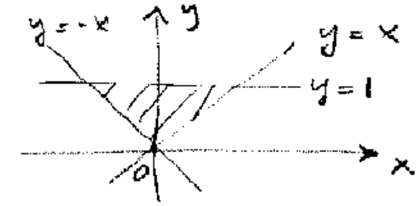
$$= \int_0^1 dy \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} x\,dx + \int_1^2 dy \int_{-\sqrt{2-y}}^{\sqrt{2-y}} x\,dx = \int_0^1 \left(\frac{x^2}{2}\right)\Big|_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dy + \int_1^2 \left(\frac{x^2}{2}\right)\Big|_{-\sqrt{2-y}}^{\sqrt{2-y}} dy =$$

$$= \int_0^1 \left(\frac{2y-y^2}{2} - \frac{2y-y^2}{2}\right) dy + \int_1^2 \left(\frac{2-y}{2} - \frac{2-y}{2}\right) dy = \int_0^1 0\,dy + \int_1^2 0\,dy = \boxed{0}$$

MATEMATIKA 2 – 3. KOLOKVIJ – 13.6.2005. – RJEŠENJA – GRUPA A

Zadatak. Prelaskom na polarne koordinate riješite integral $\iint_{(S)} \frac{x}{x^2+y^2}\,dx\,dy$, gdje je S područje omeđeno pravcima $y=x$, $y=-x$, $y=1$.

Rješenje.



$\varphi_1 = \frac{\pi}{4}$, $\varphi_2 = \frac{3\pi}{4}$

$y=1 \Rightarrow r\sin\varphi = 1 \Rightarrow r = \frac{1}{\sin\varphi}$

$\Rightarrow$ integral u polarnim koordinatama glasi:

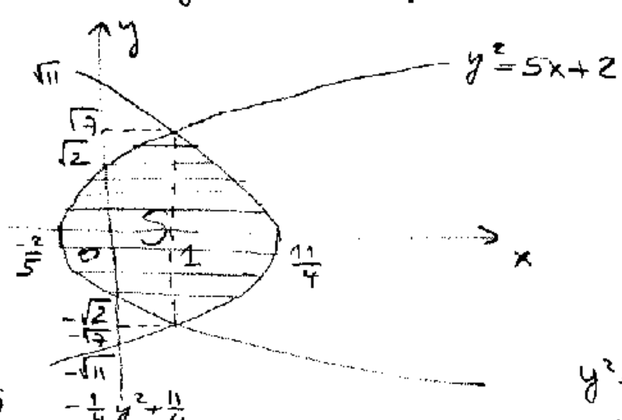
$$\int_{\pi/4}^{3\pi/4} d\varphi \int_0^{\frac{1}{\sin\varphi}} \frac{\cancel{r}\cos\varphi}{\cancel{r^2}}\,\cancel{r}\,dr = \int_{\pi/4}^{3\pi/4} (r\cos\varphi)\Big|_0^{\frac{1}{\sin\varphi}} d\varphi =$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\cos\varphi}{\sin\varphi}\,d\varphi = \left[\begin{array}{l|l} \sin\varphi = t & \varphi=\frac{\pi}{4} \Rightarrow t=\sin\frac{\pi}{4}=\frac{\sqrt2}{2} \\ \Rightarrow \cos\varphi\,d\varphi = dt & \varphi=\frac{3\pi}{4} \Rightarrow t=\sin\frac{3\pi}{4}=\frac{\sqrt2}{2}\end{array}\right] =$$

$$= \int_{\sqrt2/2}^{\sqrt2/2} \frac{dt}{t} = (\ln|t|)\Big|_{\sqrt2/2}^{\sqrt2/2} = \ln\frac{\sqrt2}{2} - \ln\frac{\sqrt2}{2} = \boxed{0}.$$

Zadatak. Koristeći dvostruki integral izračunajte površinu područja omeđenog parabolama $y^2 = 5x+2$, $y^2=-4x+11$.

Rješenje.



Računamo presjek parabola:

$5x+2 = -4x+11$

$9x = 9 \Rightarrow \underline{x=1}$

$\Rightarrow y^2 = 5\cdot1+2 = 7 \Rightarrow y = \pm\sqrt7$

$y^2 = 5x+2 \Rightarrow 5x = y^2-2 \Rightarrow x = \frac15 y^2 - \frac25$

$y^2 = -4x+11 \Rightarrow 4x = -y^2+11 \Rightarrow x = -\frac14 y^2 + \frac{11}{4}$

$$\Rightarrow P_S = \int_{-\sqrt7}^{\sqrt7}\left(\int_{\frac15 y^2-\frac25}^{-\frac14 y^2+\frac{11}{4}} 1\,dx\right)dy =$$

$$= \int_{-\sqrt7}^{\sqrt7} (x)\Big|_{\frac15 y^2-\frac25}^{-\frac14 y^2+\frac{11}{4}} dy = \int_{-\sqrt7}^{\sqrt7}\left(-\frac14 y^2+\frac{11}{4}-\frac15 y^2+\frac25\right)dy = \int_{-\sqrt7}^{\sqrt7}\left(-\frac{9}{20}y^2+\frac{63}{20}\right)dy =$$

$$= \left(-\frac{\cancel{9}^{\,3}}{20}\cdot\frac{y^3}{\cancel{3}} + \frac{63}{20}y\right)\Big|_{-\sqrt7}^{\sqrt7} = -\frac{3}{20}\left((\sqrt7)^3-(-\sqrt7)^3\right) + \frac{63}{20}\left(\sqrt7-(-\sqrt7)\right) =$$

$$= -\frac{3}{\cancel{20}_{10}}\cdot\cancel{2}\cdot7\sqrt7 + \frac{63}{\cancel{20}_{10}}\cancel{2}\sqrt7 = \sqrt7\left(\frac{63}{10}-\frac{21}{10}\right) = \frac{\cancel{42}}{\cancel{10}}\sqrt7 = \boxed{\frac{21}{5}\sqrt7}$$

Zadatak. Ispitajte konvergenciju reda $\sum_{n=1}^{\infty} \frac{n+5}{n^2+3n}$.

Rješenje. Označimo $b_n = \frac{n+5}{n^2+3n}$ opći član reda i usporedimo zadani red s $\sum_{n=1}^{\infty} \frac{1}{n}$, harmonijskim redom, čiji opći član označimo s $a_n = \frac{1}{n}$. Svi članovi $a_n, b_n, n \in \mathbb{N}$, su nenegativni i vrijedi

$a_n \leq b_n, \ \forall n \in \mathbb{N}$, jer je: $\frac{1}{n} \leq \frac{n+5}{n^2+3n}$

$$\Longleftrightarrow \frac{1}{n} \leq \frac{1}{n} \cdot \frac{n+5}{n+3} \ /\cdot n$$

$$\Longleftrightarrow 1 \leq \frac{n+5}{n+3} \ /\cdot(n+3)$$

$$\Longleftrightarrow n+3 \leq n+5$$ očito, pa

prema 1. kriteriju uspoređivanja iz činjenice da harmonijski red divergira zaključujemo da divergira i red $\sum_{n=1}^{\infty} \frac{n+5}{n^2+3n}$.

Zadatak. Razvijte u red oko nule funkciju $f(x) = \frac{5}{6x^2-5x+1}$ te nađite područje konvergencije tog reda.

Rješenje: $f(x) = \frac{5}{\underbrace{6x^2-5x+1}_{=0}} = \frac{5}{6(x-\frac{1}{3})(x-\frac{1}{2})} = \frac{5}{(3x-1)(2x-1)} = \frac{5}{(1-2x)(1-3x)}$

$$\Rightarrow x_{1,2} = \frac{5 \pm \sqrt{25-24}}{12} = \frac{5 \pm 1}{12} \Rightarrow x_1 = \frac{1}{3}, x_2 = \frac{1}{2}$$

$\Rightarrow$ radimo rastav na parcijalne razlomke:

$$\frac{5}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x} = \frac{A-3Ax+B-2Bx}{(1-2x)(1-3x)} \Rightarrow \begin{array}{l} A+B=5 \ /\cdot 2 \\ -3A-2B=0 \\ \hline -A=10 \Rightarrow A=-10 \\ B=15 \end{array}$$

$$\Rightarrow f(x) = \frac{15}{1-3x} - \frac{10}{1-2x} =$$

$$= 15 \cdot \sum_{n=0}^{\infty} (3x)^n - 10 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} \underbrace{(15 \cdot 3^n - 10 \cdot 2^n) x^n}_{a_n}$$ je traženi razvoj

Područje konvergencije:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{15 \cdot 3^{n+1} - 10 \cdot 2^{n+1}}{15 \cdot 3^n - 10 \cdot 2^n} \cdot \frac{\cancel{x^{n+1}}^{\,x}}{\cancel{x^n}} \right| \ [: 3^{n+1}] = \lim_{n\to\infty} \left| \frac{15 - 10 \cdot \left(\frac{2}{3}\right)^{n+1} \to 0}{15 \cdot \frac{1}{3} \cdot 1 - 10 \cdot \frac{1}{2} \left(\frac{2}{3}\right)^{n+1} \to 0} \cdot x \right|$$

$$= \frac{\cancel{15}}{\cancel{15} \cdot \frac{1}{3}} |x| = 3|x| < 1 \Longleftrightarrow |x| < \frac{1}{3} \Longleftrightarrow \boxed{x \in \langle -\frac{1}{3}, \frac{1}{3} \rangle}$$

Još treba vidjeti što se događa u rubovima:

$\underline{X = -\frac{1}{3}}$ $\rightarrow$ red ovdje glasi $\sum_{n=0}^{\infty} (15\cdot 3^n - 10\cdot 2^n)\cdot\left(-\frac{1}{3}\right)^n =$

$$= \sum_{n=0}^{\infty} \frac{15\cdot 3^n - 10\cdot 2^n}{3^n}\cdot(-1)^n = \sum_{n=0}^{\infty}\left(15 - 10\cdot\left(\tfrac{2}{3}\right)^n\right)\cdot(-1)^n$$

Kako ne vrijedi $\lim_{n\to\infty} a_n = 0$, što je nužan uvjet konvergencije, red ne konvergira i $X = -\frac{1}{3}$ se ne nalazi u intervalu konvergencije.

$\underline{X = \frac{1}{3}}$ $\rightarrow$ red glasi $\sum_{n=0}^{\infty}(15\cdot 3^n - 10\cdot 2^n)\cdot\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{15\cdot 3^n - 10\cdot 2^n}{3^n} =$

$= \sum_{n=0}^{\infty}\left(15 - 10\left(\frac{2}{3}\right)^n\right)$ $\rightarrow$ iz istog razloga kao gore red divergira

$\Rightarrow$ područje konvergencije je $\underline{\left\langle -\frac{1}{3}, \frac{1}{3}\right\rangle}$.

Zadatak. Riješite Cauchyjev problem $y'' - 9y' + 18y = x + 1$, $y(0) = 0$, $y'(0) = 1$.

Rješenje. Rješavamo karakt. jednadžbu $\lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 6$

$\Rightarrow y_H = C_1 e^{3x} + C_2 e^{6x}$ je homogeno rješenje.

Partikularno rješenje tražimo u obliku $y_P = Ax + B \Rightarrow y_P' = A,\ y_P'' = 0$.

Uvrštavanje u jednadžbu daje $-9A + 18(Ax + B) = x + 1$

$$\Rightarrow \begin{array}{l} 18A = 1 \\ -9A + 18B = 1 \ /\cdot 2 \quad + \\ \hline 36B = 3 \Rightarrow B = \frac{1}{12},\ A = \frac{1}{18} \end{array}$$

$\Rightarrow y_1 = \frac{1}{18}x + \frac{1}{12}$

$\Rightarrow y = y_H + y_P = C_1 e^{3x} + C_2 e^{6x} + \frac{1}{18}x + \frac{1}{12}$

Početni uvjeti daju $y(0) = \boxed{C_1 + C_2 + \frac{1}{12} = 0}$ $\ /\cdot(-3)$

$y' = 3C_1 e^{3x} + 6C_2 e^{6x} + \frac{1}{18} \rightarrow y'(0) = \boxed{3C_1 + 6C_2 + \frac{1}{18} = 1}$ $\quad +$

$$3C_2 - \frac{\cancel{3}}{\cancel{12}_4} + \frac{1}{18} = 1$$

$$3C_2 = 1 + \frac{1}{4} - \frac{1}{18} = \frac{36 + 9 - 2}{36} = \frac{43}{36} \Rightarrow \boxed{C_2 = \frac{43}{108}}$$

$$C_1 = -\frac{1}{12} - \frac{43}{108} \Rightarrow \boxed{C_1 = \frac{-9-43}{108} = \frac{-52}{108}}$$

$$\Rightarrow \boxed{y = \frac{-52}{108}e^{3x} + \frac{43}{108}e^{6x} + \frac{1}{18}x + \frac{1}{12}}$$