MATEMATIKA 2

PISMENI ISPITI 2003. – RJEŠENJA

10. svibnja
1. srpnja

MAT II, 10.05.2003.

(1) Odredite domenu fkcije

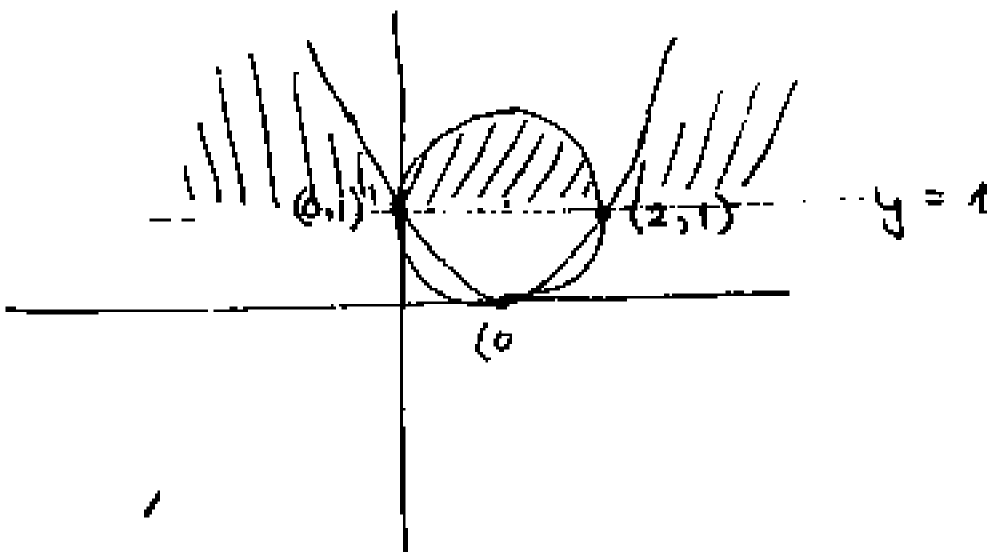
$$f(x,y) = \sqrt{((x-1)^2+(y-1)^2-1)((x-1)^2-y)} + \ln(y-1)$$

Rj zbog $\ln(y-1)$ odmah imamo $y-1>0 \Rightarrow y>1$
Rješavamo korjen. Dvije su mogućnosti:

$(x-1)^2+(y-1)^2-1 \le 0$ ili $(x-1)^2+(y-1)^2-1 \geq 0$

$(x-1)^2-y \le 0$ $\qquad$ $(x-1)^2-y \geq 0$

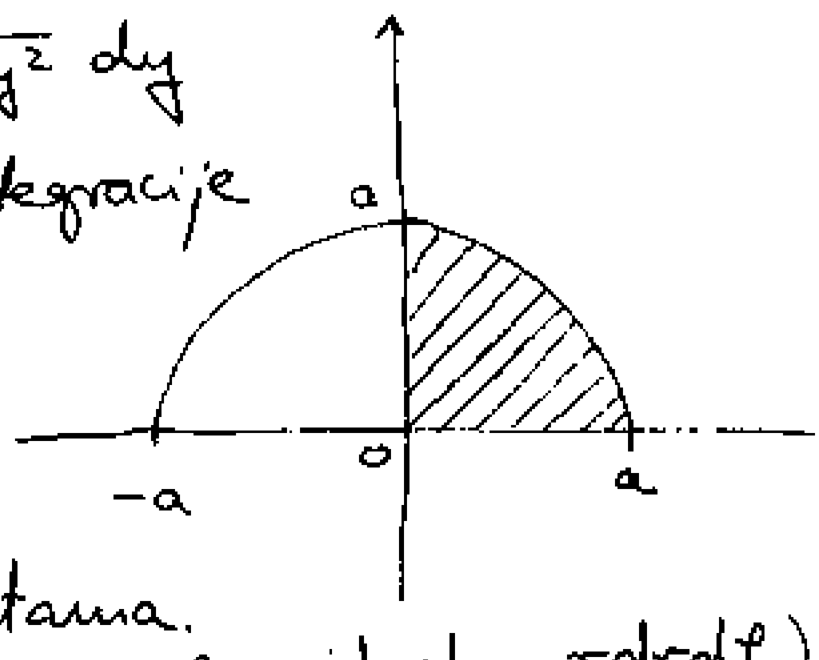
prva krivulja je kružnica s centrom u $(1,1)$ a druga parabola sa tjemenom u $x=1$ pa imamo



(2) Prijelazom na polarne koordinate izračunajte:

$$\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy$$

Rj Crtamo područje integracije



pa u polarnim koordinatama
imamo ($x=r\cos\varphi$, $y=r\sin\varphi$, $dxdy=rdrd\varphi$)

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^a r\cdot r dr = \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r^2 dr = \int_0^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \Big|_0^a \right) d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} a^3 d\varphi = \frac{1}{3} a^3 \frac{\pi}{2} = \frac{a^3 \pi}{6}$$

③ Pokažite da ravnine tangencijalne na plohu $z = \frac{y^2}{x}$ u proizvoljnoj točki $M(x_0, y_0, z_0)$ prolaze ishodištem koord. sistema.

Rj. Tražimo jednadžbu tang. ravnine u točki $M(x_0, y_0, z_0)$

To je:

$$z - z_0 = \frac{\partial z}{\partial x}\Big|_M (x - x_0) + \frac{\partial z}{\partial y}\Big|_M (y - y_0)$$

Imamo: $\frac{\partial z}{\partial x} = -\frac{y^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}\Big|_M = -\frac{y_0^2}{x_0^2}$

$$\frac{\partial z}{\partial y} = \frac{2y}{x} \Rightarrow \frac{\partial z}{\partial y}\Big|_M = \frac{2y_0}{x_0}$$

pa ravnina glasi:

$$z - z_0 = -\frac{y_0^2}{x_0^2}(x - x_0) + \frac{2y_0}{x_0}(y - y_0)$$

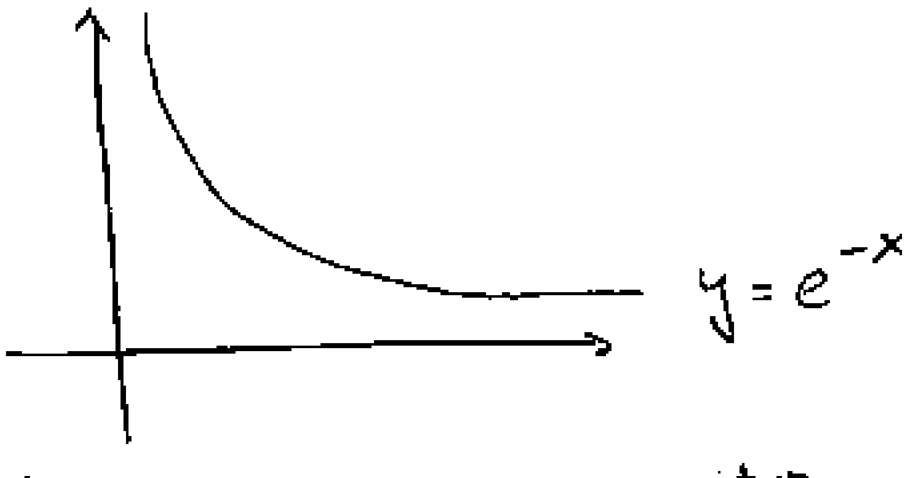
Uvstavamo ishodište, $x = 0, y = 0, z = 0$

$$\Rightarrow -z_0 = \frac{y_0^2}{x_0} - \frac{2y_0^2}{x_0} = -\frac{y_0^2}{x_0}$$

tj. $z_0 = \frac{y_0^2}{x_0}$ a to vrijedi jer $M(x_0, y_0, z_0)$ leži na plohi $z = \frac{y^2}{x}$. Stoga je „ishodište u tang. ravnini

④ Nađite površinu plohe nastalu rotacijom oko osi OX luka krivulje $y = e^{-x}$ od $x = 0$ do $x = +\infty$

Rj



$$P = 2\pi \int_0^{+\infty} y\sqrt{1 + y'^2}\,dx = 2\pi \int_0^{+\infty} e^{-x}\sqrt{1 + e^{-2x}}\,dx = \left| \begin{array}{l} e^{-x} = t \\ -e^{-x}dx = dt \\ t_1 = e^{-0} = 1 \\ t_2 = e^{-\infty} = 0 \end{array} \right.$$

$$= 2\pi \int_1^0 -\sqrt{1 + t^2}\,dt = 2\pi \int_0^1 \sqrt{1 + t^2}\,dt =$$

$$= 2\pi \left(\frac{t}{2}\sqrt{t^2 + 1} + \frac{1}{2}\ln|t + \sqrt{t^2 + 1}| \Big|_0^1 \right) =$$

$= 2\pi\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2})\right) = \left(\sqrt{2} + \ln(1+\sqrt{2})\right)\pi$

(5) Riješite Cauchyev problem:

$$y y'' + y'^2 = 1, \quad y(0) = 1 \quad i \quad y'(0) = 1$$

Rj. Diferencijalna jednadžba ne sadrži eksplicitno x pa možemo upotrijebiti supstituciju

$$y' = p, \quad y'' = p\frac{dp}{dy}$$

$$\Rightarrow \quad y \cdot p\frac{dp}{dy} + p^2 = 1$$

tj. $\quad pp' + \frac{p^2}{y} = \frac{1}{y} \Rightarrow p' + \frac{p}{y} = \frac{1}{py}$ za $p \neq 0$

stavimo $p = uv$

$$\Rightarrow u'v + uv' + \frac{uv}{y} = \frac{1}{uvy}$$

$$\Rightarrow v\left(u' + \frac{u}{y}\right) + uv' = \frac{1}{uvy} \quad (*)$$

Sada rješavamo $u' + \frac{u}{y} = 0 \Rightarrow \frac{du}{u} = -\frac{dy}{y}$

$$\Rightarrow \ln u = -\ln y \Rightarrow u = \frac{1}{y}$$

Uvrštavamo to u $(*)$. Slijedi

$$\frac{v'}{y} = \frac{1}{\frac{1}{\cancel{y}} v \cancel{y}} = \frac{1}{v} \Rightarrow v\,dv = y\,dy$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{1}{2}y^2 + \tilde{C}_1 \Rightarrow v^2 = y^2 + C_1 \Rightarrow$$

$$\Rightarrow v = \sqrt{y^2 + C_1} \quad \text{pa} \quad p = uv = \frac{\sqrt{y^2+C_1}}{y} = \sqrt{1 + \frac{C_1}{y}}$$

tj. $\quad y' = \sqrt{1 + \frac{C_1}{y}}$

$$y'(0) = \sqrt{1 + \frac{C_1}{y(0)}} \quad \text{pa, jer } y(0) = y'(0) = 1$$

slijedi

$$1 = \sqrt{1 + \frac{C_1}{1}} \Rightarrow C_1 = 0 \Rightarrow y' = 1$$

tj. $dy = dx \Rightarrow y = x + C_2$

$y(0) = 1 \Rightarrow C_2 = 1 \Rightarrow y = x + 1$ je rješenje.

Ostaje još provjeriti slučaj $p = 0 \Rightarrow \frac{dy}{dx} = y' = 0$

No, $y'(0) = 1$ pa to nije moguće.

Stoga je jedino rješenje $y = x + 1$.

MATEMATIKA II - 1.07.2003.

1. Odredite domenu funkcije $f(x,y)=\sqrt{\arcsin(|x|-|y|)}$

Rj. zbog $\sqrt{\ }$ $\Rightarrow$ $\arcsin(|x|-|y|)\geq 0$

$\arcsin t: [-1,1] \longrightarrow [-\frac{\pi}{2},\frac{\pi}{2}]$ i iz toga slijedi

$0\leq |x|-|y| \leq 1$

Rješavamo prvu nejednakost: $0\leq |x|-|y|$

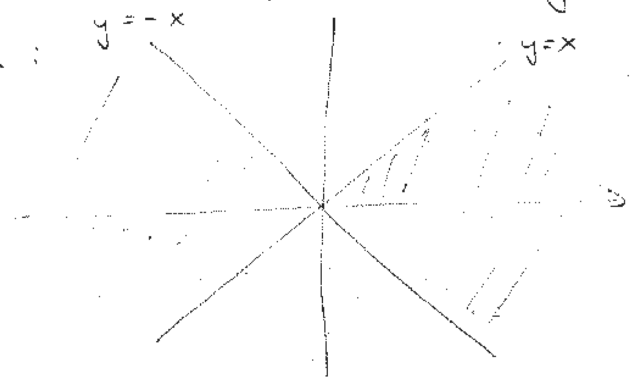
I kvadrant, $x\geq 0, y\geq 0 \Rightarrow y\leq x$

II —"— $x\leq 0, y\geq 0 \Rightarrow y\leq -x$

III —"— $x\leq 0, y\leq 0 \Rightarrow y\geq x$

IV —"— $x\geq 0, y\leq 0 \Rightarrow y\geq -x$

Sve skupa:



Druga nejednakost $|x|-|y|\leq 1$

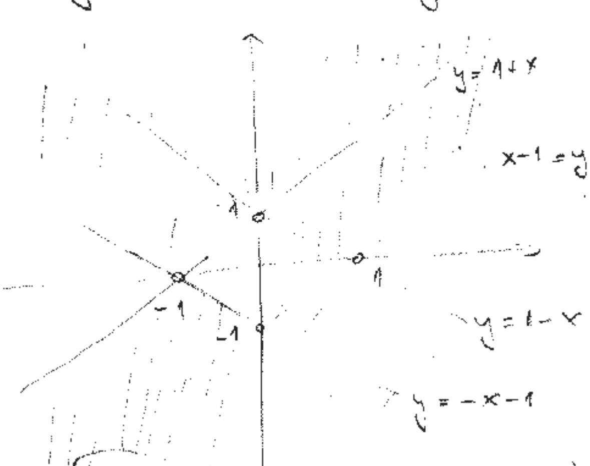
I kvadrant, $x\geq 0; y\geq 0 \Rightarrow x-y\leq 1 \Rightarrow y\geq x-1$

II —"— $x\leq 0, y\geq 0 \Rightarrow -x-y\leq 1 \Rightarrow y\geq -x-1$

III —"— $x\leq 0; y\leq 0 \Rightarrow -x+y\leq 1 \Rightarrow y\leq 1+x$

IV —"— $x\geq 0, y\leq 0 \Rightarrow x+y\leq 1 \Rightarrow y\leq 1-x$

Sve skupa



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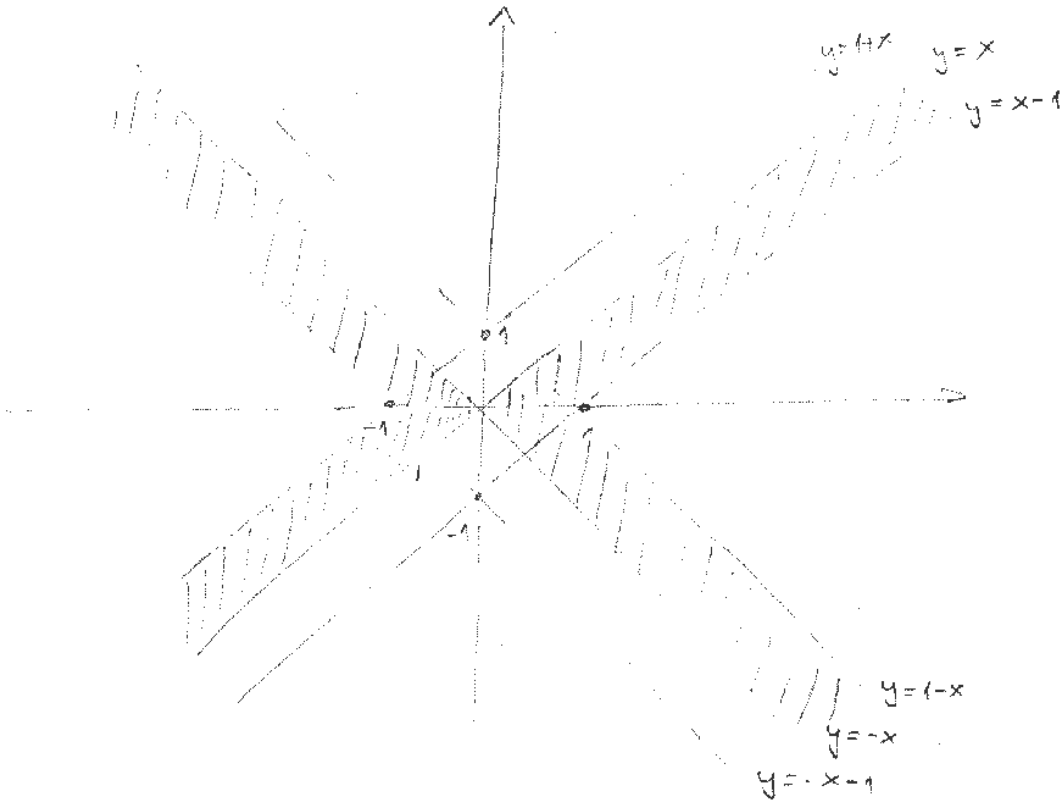
Tražimo presjek sa prvim grafom:



2. Od svih kvadera volumena 27 nađite onaj koji ima najmanje oplošje.

Rj. volumen: $V = abc = 27$

oplošje: $O = 2ab + 2ac + 2bc$

Iz volumena $c = \frac{27}{ab}$ pa imamo za oplošje

$$O(a,b) = 2ab + 2\cancel{a}\frac{27}{\cancel{a}b} + 2\cancel{b}\frac{27}{a\cancel{b}} = 2ab + \frac{54}{b} + \frac{54}{a}$$

Tražimo minimum:

$$\frac{\partial O}{\partial a} = 2b - \frac{54}{a^2}, \quad \frac{\partial O}{\partial b} = 2a - \frac{54}{b^2}$$

$$\frac{\partial O}{\partial a} = \frac{\partial O}{\partial b} = 0 \Rightarrow 2b = \frac{54}{a^2}; \quad 2a = \frac{54}{b^2}$$

$$\Rightarrow b = \frac{27}{a^2}; \quad a = \frac{27}{b^2}$$

$$\Rightarrow b = \frac{27}{\frac{27^2}{b^4}} \Rightarrow b = \frac{b^4}{27} \Rightarrow 27 = b^3 \Rightarrow b = 3$$

$$\Rightarrow a = 3$$

pa je to kandidat za ekstrem. Ispitujemo druge derivacije:

$$\frac{\partial^2 O}{\partial a^2} = 2\cdot\frac{54}{a^3}, \quad \frac{\partial^2 O}{\partial b^2} = 2\cdot\frac{54}{b^3}, \quad \frac{\partial^2 O}{\partial a \partial b} = 2$$

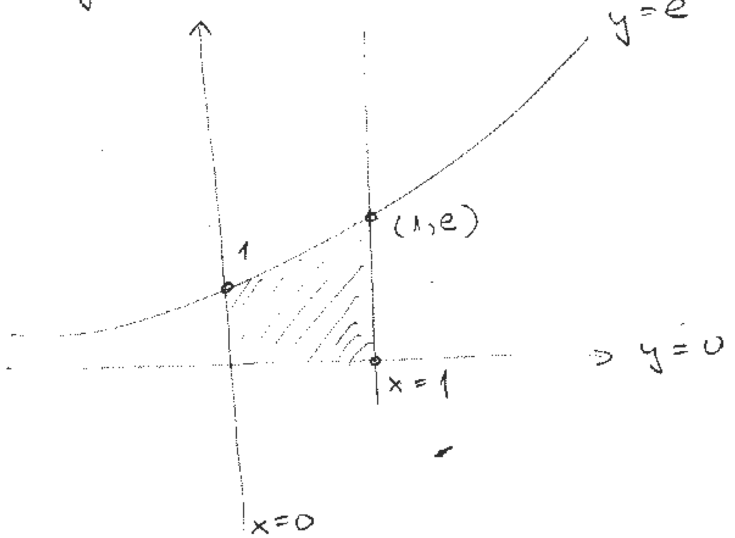
$$A = 2\cdot\frac{54}{27} = 4, \quad B = 2, \quad C = \frac{2\cdot 54}{27} = 4$$

$$\Rightarrow \quad \Delta = AC - B^2 = 16 - 4 = 12 > 0 \quad \text{pa je}$$

tu stvarno ekstrem, $A > 0 \Rightarrow$ to je minimum.

3 Nađite volumen tijela nastalog rotacijom plohe omeđene sa krivuljom $y = e^x$ i pravcima $x=0$, $x=1$ te $y=0$ oko osi OX

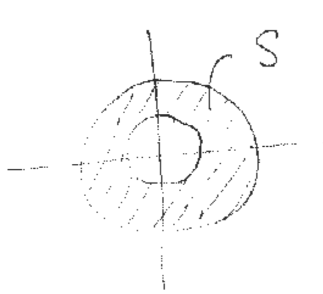
Rj:



$$V = \pi\int_a^b y^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{\pi}{2}e^{2x}\Big|_0^1 = \frac{\pi}{2}e^2 - \frac{\pi}{2}$$

4 Izračunajte integral $\iint_S y\,dx\,dy$ ako je S kružni vijenac omeđen kružnicama polumjera $r=1$ i $R=2$ sa središtem u ishodištu

Rj:



Prelazimo na polarne koordinate, kut je očito od 0 do 2π a r ide od 1 do 2 pa imamo

$$\iint_{(S)} y\,dx\,dy = \int_0^{2\pi} d\varphi \int_1^2 r\sin\varphi \; r\,dr = \int_0^{2\pi} d\varphi \int_1^2 r^2 \sin\varphi \, dr =$$

$$= \int_0^{2\pi} \left(\frac{1}{3} r^3 \Big|_1^2 \right) \sin\varphi \, d\varphi = \frac{7}{3} \int_0^{2\pi} \sin\varphi \, d\varphi =$$

$$= \frac{7}{3} \left(-\cos\varphi\right) \Big|_0^{2\pi} = \frac{7}{3} \cdot (-1+1) = 0$$

5 Riješite dif. jednadžbu $y'' - 2y' + y = 2e^x$

Rj. Rješavamo prvo homogeni dio

$$y'' - 2y' + y = 0$$

karakt. jedn. $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 1$$

pa je rješenje $\quad y_0 = e^x (C_1 + C_2 x)$

Tražimo partikularno rješenje: $f(x) = 2e^x$

$$\Rightarrow a = 1, \quad P_n(x) = 2$$

a je korijen karakt. jedn. pa rješenje tražimo u obliku $\quad Y = x^2 e^x \cdot C$

$$Y' = 2x e^x \cdot C + x^2 e^x \cdot C$$

$$Y'' = 2e^x \cdot C + 2x e^x \cdot C + 2x e^x \cdot C + x^2 e^x \cdot C =$$

$$= 2Ce^x + 4Cxe^x + Cx^2 e^x$$

$$\Rightarrow \cancel{Cx^2e^x} + \cancel{4Cxe^x} + 2Ce^x - \cancel{4xe^x \cdot C} - \cancel{2x^2e^x \cdot C} + \cancel{Cx^2e^x} = 2e^x$$

$$\Rightarrow 2Ce^x = 2e^x \Rightarrow C = 1$$

pa $\quad Y = x^2 e^x$

i sve skupa

$$y = y_0 + Y = e^x (C_1 + C_2 x) + x^2 e^x$$