MATEMATIKA 2

PISMENI ISPITI 2004. – RJEŠENJA

6. srpnja
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RJEŠENJE - MAT II 6.07.2004.

① Odredite domenu funkcije $f(x,y)=\sqrt{\sin|x+y|}+\ln(4-(x+y)^2)$ i skicirajte je u koordinatnoj ravnini.

Rj. zbog $\sqrt{}$: $\sin|x+y|\geq 0 \Rightarrow |x+y|\in[2k\pi,(2k+1)\pi]$; $k\in\mathbb{Z}$

zbog ln : $4-(x+y)^2>0 \Rightarrow (x+y)^2<4 \;\sqrt{}$

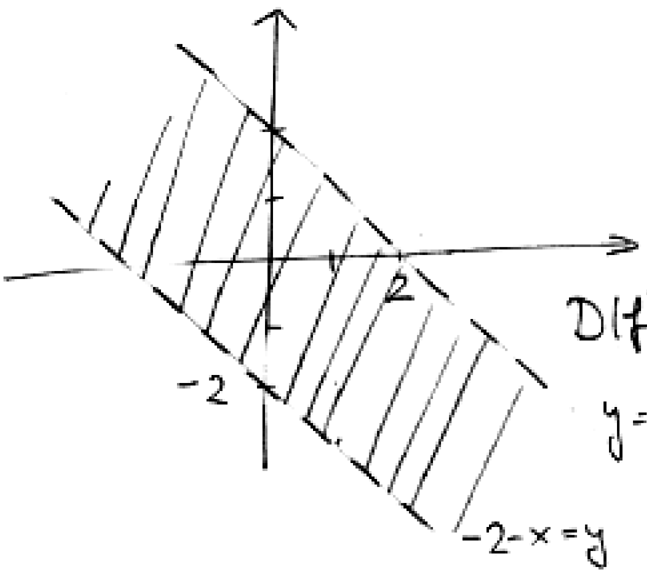
$\Rightarrow |x+y|<2$

Presjek $\sqrt{}$ i ln : $0\leq|x+y|<2$

$\Rightarrow -2<x+y<2$

$-2<x+y$; $x+y<2$

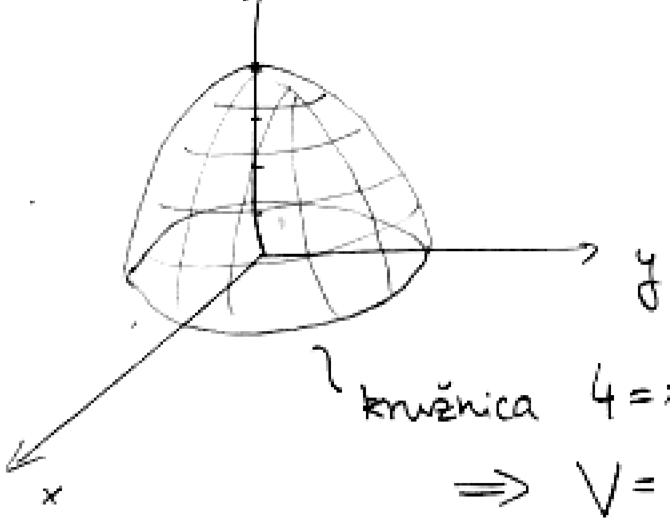
$\Rightarrow -2-x<y$; $y<2-x$



D(f) (bez pravaca)

② Nađite volumen tijela određenog sa $z=4-x^2-y^2$; $z\geq 0$

Rj.



kružnica $4=x^2+y^2$

$\Rightarrow V=\iint\limits_{D}(4-x^2-y^2)\,dx\,dy$

gdje je D krug radijusa dva sa centrom u $(0,0)$.
Prelazimo na polarne koordinate.

$x = r\cos\varphi \quad y = r\sin\varphi \qquad x^2+y^2=4 \Rightarrow r^2=4$

$\Rightarrow r=2$

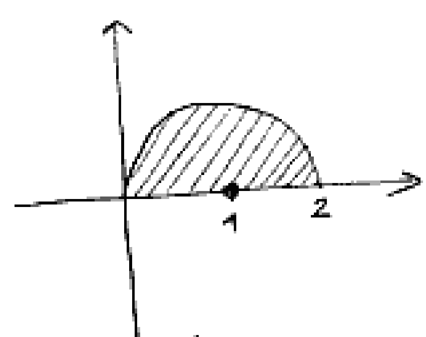
$$\Rightarrow V=\iint_D (4-x^2-y^2)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^2 (4-r^2)\,r\,dr =$$

$$=\int_0^{2\pi} d\varphi \int_0^2 (4r-r^3)\,dr = \int_0^{2\pi} \left(2r^2-\frac{1}{4}r^4 \Big|_0^2\right) d\varphi =$$

$$=\int_0^{2\pi} \left(2\cdot 4-\frac{1}{\cancel{4}}\cdot \overset{4}{\cancel{16}}-0-0\right) d\varphi = \int_0^{2\pi} 4\,d\varphi = 8\pi$$

③ Izračunati $\iint_D \sqrt{1-x^2-y^2}\,dx\,dy$ gdje D područje omeđeno sa x-osi i kružnicom $(x-1)^2+y^2=1$.

Rj: područje D:



$x^2-2x+y^2=0$

$r^2-2r\cos\varphi=0$

$r=2\cos\varphi$

jedn. kružnice u polarnim koord.

Prelazimo na polarne koordinate:

$$\iint_D \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \sqrt{1-r^2}\,r\,dr =$$

$$=\int_0^{\pi/2} -\frac{2}{3}\cdot\frac{1}{2}(1-r^2)^{\frac{3}{2}} \Big|_0^{2\cos\varphi} d\varphi = -\frac{1}{3}\int_0^{\pi/2} (1-4\cos^2\varphi)^{\frac{3}{2}}\,d\varphi$$

i to je eliptički integral koji se može ovako ostaviti.

(4) Razvijte u Taylorov red oko $x_0 = 3\pi$ funkciju $f(x) = \cos x$ i odredite područje konv. tog reda.

Rj

Znamo da je (po tablicama za razvoj od $\cos x$):

$$\cos(x-3\pi) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-3\pi)^{2n}}{(2n)!}, \quad -\infty < x < +\infty$$

Dalje imamo:

$$\cos(x-3\pi) = \cos x \cos 3\pi + \sin x \underbrace{\sin 3\pi}_{0} = -\cos x$$

Stoga

$$f(x) = \cos x = -\cos(x-3\pi) = -\sum_{n=0}^{\infty} (-1)^n \frac{(x-3\pi)^{2n}}{(2n)!} =$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-3\pi)^{2n}}{(2n)!}$$

i to je traženi razvoj koji konvergira za svako $x \in \mathbb{R}$.

(5) Riješite Cauchyev problem $y'' - y' = e^x$, $y(0) = 1$, $y'(0) = 0$.

Rj karakt. jedn. $\lambda^2 - \lambda = 0 \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 1$

$\Rightarrow$ homogeno rješ. je $\quad y_0 = C_1 e^{0 \cdot x} + C_2 e^{1 \cdot x} =$

$= C_1 + C_2 e^x$

Tražimo partikularno.

$f(x) = e^x = e^{ax} P_n(x) \Rightarrow a = 1, \; P_n(x) = 1$ pa je

Y oblika $\quad Y(x) = x e^x \cdot A$

$Y'(x) = Ae^x + Axe^x, \quad Y''(x) = Ae^x + Ae^x + Axe^x$

$\Rightarrow 2Ae^x + Axe^x - Ae^x - Axe^x = e^x \Rightarrow \quad Ae^x = e^x$

pa $A = 1 \Rightarrow Y(x) = xe^x$

$\Rightarrow y = C_1 + C_2 e^x + xe^x \Rightarrow y' = C_2 e^x + xe^x + e^x$

$y(0) = 1 \Rightarrow C_1 + C_2 = 1, \; y'(0) = 0 \Rightarrow C_2 + 1 = 0 \Rightarrow C_2 = -1$

$C_1 = 2$

MAT II - RJEŠENJA 13/07/04

① Odredite domenu: $f(x,y)=\sqrt{\log_{x^2+y^2}(5-x^2-y^2)}$ i skicirajte je u koordin. ravnini.

Rj. zbog $\sqrt{}$: $\log_{x^2+y^2}(5-x^2-y^2)\geq 0$

a) $0<x^2+y^2<1$

$\Rightarrow (x^2+y^2)^{\log_{x^2+y^2}(5-x^2-y^2)} \leq (x^2+y^2)^0$

$\Rightarrow 5-x^2-y^2\leq 1$

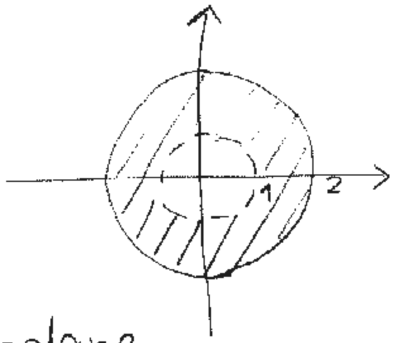
$4\leq x^2+y^2$ pa tu nema ničeg

b) $1<x^2+y^2$

$\Rightarrow 5-x^2-y^2\geq 1$

$4\geq x^2+y^2$

$D(f)=\{(x,y)\mid 1<x^2+y^2\leq 4\}$



② Nađite lokalne ekstreme fkcije zadane sa $x^2+2x+y^2+z^2-z=0$

Rj. $F(x,y,z)=x^2+2x+y^2+z^2-z$

$\frac{\partial z}{\partial x}(x,y)=-\frac{F_x}{F_z}=-\frac{2x+2}{2z-1}$

$\frac{\partial z}{\partial y}(x,y)=-\frac{F_y}{F_z}=-\frac{2y}{2z-1}$

$\Rightarrow \frac{\partial z}{\partial x}(x,y)=0 \Rightarrow x=-1;\ \frac{\partial z}{\partial y}(x,y)=0\Rightarrow y=0$

$z(-1,0):\quad 1-2+0+z^2-z=0$

$\Rightarrow z^2-z-1=0$

$z=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt5}{2}$

pa $2z-1\neq 0$ i to je dobar kandidat za lokalni ekstrem.

$$\frac{\partial^2 z}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(-\frac{2x+2}{2z-1}\right) = -2\frac{\partial}{\partial x}\left(\frac{x+1}{2z-1}\right) =$$

$$= -2\,\frac{2z-1-(x+1)2\cdot\frac{\partial z}{\partial x}}{(2z-1)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}(-1,0) = -2\,\frac{2z(-1,0)-1}{(2z(-1,0)-1)^2} = \frac{-2}{2z(-1,0)-1} = A$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{2y}{(2z-1)}\right) = -2\frac{\partial}{\partial y}\left(\frac{y}{2z-1}\right) = \frac{-2}{2z(-1,0)-1} = C$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{2y}{2z-1}\right) = -2\frac{\partial}{\partial x}\left(\frac{y}{2z-1}\right) = -2\cdot\frac{0(2z-1)-y\cdot 2\frac{\partial z}{\partial x}}{(2z-1)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y}(-1,0) = 0$$

$$\Rightarrow AC - B^2 = \frac{4}{(2z(-1,0)-1)^2} - 0 > 0$$

pa je tu lok. ekstrem

Ako $z = \frac{1+\sqrt{5}}{2}$

$\Rightarrow A = \frac{-2}{1+\sqrt{5}-1} < 0$ pa je tu lok. max.

i za $z = \frac{1-\sqrt{5}}{2}$

$A = \frac{-2}{1-\sqrt{5}-1} > 0$ pa je lok. min.

3) Izračunati dvostruki integral $\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy$

gdje je D omeđen kružnicama $x^2+(y-1)^2=1$
$x^2+y^2=1$

Rj. Skiciramo D: $x^2+(y-1)^2=1$

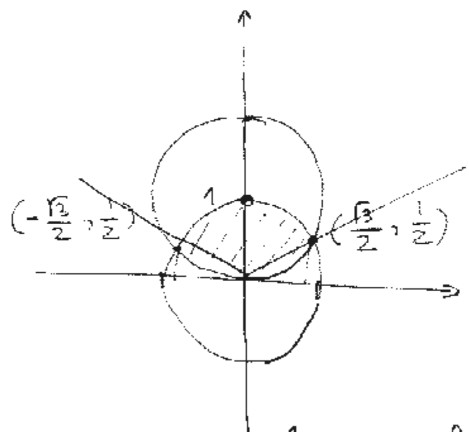
$x^2+y^2=1$

$\Rightarrow\ y^2-(y-1)^2=0$

$y^2-y^2+2y-1=0$

$y=\frac{1}{2}$

$x^2=\frac{3}{4} \Rightarrow x=\frac{\pm\sqrt{3}}{2}$



kut od $\frac{\pi}{6}$

Prelazimo na polarne koordinate: $r^2\cos^2\varphi+r^2\sin^2\varphi-$

$-2r\sin\varphi=0$

$\Rightarrow r^2=2r\sin\varphi$

$r=2\sin\varphi$ je jednadžba od $x^2+(y-1)^2=1$

a $x^2+y^2=1$ glasi $r=1$.

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi} r\sqrt{r^2}\,dr+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}d\varphi\int_0^1 r\sqrt{r^2}\,dr$$

$$+\int_{\frac{5\pi}{6}}^{\pi}d\varphi\int_0^{2\sin\varphi} r\sqrt{r^2}\,dr=\int_0^{\frac{\pi}{6}}\frac{1}{3}r^3\Big|_0^{2\sin\varphi}d\varphi+$$

$$+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\frac{1}{3}r^3\Big|_0^1 d\varphi+\int_{\frac{5\pi}{6}}^{\pi}\frac{1}{3}r^3\Big|_0^{2\sin\varphi}d\varphi=$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{6}}\sin^3\varphi\,d\varphi+\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}d\varphi+\frac{8}{3}\int_{\frac{5\pi}{6}}^{\pi}\sin^3\varphi\,d\varphi=$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{6}}(\sin\varphi-\cos^2\sin\varphi)\,d\varphi+\frac{1}{3}\cdot\frac{4}{6}\pi+\frac{8}{3}\int_{\frac{5\pi}{6}}^{\pi}(\sin\varphi-\cos^2\varphi\sin\varphi)$$

$\{\sin^3\varphi=\sin^2\varphi\sin\varphi=(1-\cos^2\varphi)\sin\varphi\}$

$$= \frac{8}{3}\left(-\cos\varphi + \frac{1}{3}\cos^3\varphi \Big|_0^{\frac{\pi}{6}}\right) + \frac{2}{9}\pi + \frac{8}{3}\left(\frac{1}{3}\cos^3\varphi - \cos\varphi \Big|_{\frac{5\pi}{6}}^{\pi}\right)$$

$$= \frac{8}{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 + 1 - \frac{1}{3}\right) + \frac{2}{9}\pi + \frac{8}{3}\left(-\frac{1}{3} + 1 + \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{8}{3}\left(-\sqrt{3} + \frac{1}{3}\frac{(\sqrt{3})^3}{4} + 2 - \frac{2}{3}\right) + \frac{2}{9}\pi$$

4) Izračunajte približno $\sqrt{5.7 + \sqrt[3]{1-9.2}}$

Rj. $f(x,y) = \sqrt{x + \sqrt[3]{1-y}}$

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\,\Delta y$$

$x_0 = 6 \qquad x_0 + \Delta x = 5.7 \Rightarrow \Delta x = -0.3$

$y_0 = 9 \qquad y_0 + \Delta y = 9.2 \Rightarrow \Delta y = 0.2$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+\sqrt[3]{1-y}}} \; ; \quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x+\sqrt[3]{1-y}}} \cdot \frac{1}{3}(1+y)^{-\frac{2}{3}}(-1)$$

$$f(x_0,y_0) = \sqrt{6+\sqrt[3]{1-9}} = \sqrt{6+\sqrt[3]{-8}} = \sqrt{6-2} = \sqrt{4} = 2$$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \frac{1}{2\sqrt{6+\sqrt[3]{1-9}}} = \frac{1}{2\sqrt{6-2}} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \frac{1}{4}\cdot\frac{1}{3}(1-9)^{-\frac{2}{3}}(-1) = -\frac{1}{12}\,\frac{1}{\sqrt[3]{(-8)^2}} =$$

$$= -\frac{1}{12}\,\frac{1}{\sqrt[3]{64}} = -\frac{1}{12\cdot 4} = -\frac{1}{48}$$

$$f(5.7, 9.2) \approx 2 + \frac{1}{4}\cdot(-0.3) - \frac{1}{48}\cdot 0.2 =$$

$$= 2 - \frac{3}{40} - \frac{2}{480} = 2 - \frac{3}{40} - \frac{1}{240} = 2 - \frac{18+1}{240} =$$

$$= 2 - \frac{19}{240}$$

⑤ Zadana je $f(x)=\sum_{n=0}^{+\infty}\left(\frac{x}{3}\right)^n$. Naći $D(f)$, $f(1)$

i $f^{(100)}(0)$.

Rj. $D(f)$ = područje konv. reda $\Rightarrow \left|\frac{x}{3}\right|<1 \Rightarrow |x|<3$

$\Rightarrow D(f)=\{x \mid -3<x<3\}$

$$f(1)=\sum_{n=0}^{+\infty}\left(\frac{1}{3}\right)^n=\frac{1}{1-\frac{1}{3}}=\frac{3}{3-1}=\frac{3}{2}$$

to je geom. red

Jer. $$f(x)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{+\infty}\frac{x^n}{3^n}$$

$$\Rightarrow \frac{f^{(100)}(0)}{100!}=\frac{1}{3^{100}} \Rightarrow f^{(100)}(0)=\frac{100!}{3^{100}}$$

MAT II - RJESENJA 9/09/2004

① Odrediti domenu f-kcije
$$f(x,y)=\sqrt{y^2-16x^2}+\ln\left((x+y)^2-5(x+y)+6\right)$$
i skicirati ju.

Rj. zbog $\sqrt{}$: $y^2-16x^2\geq 0 \Rightarrow y^2\geq 16x^2$

tj. $|y|\geq|4x|$

I kvadrant : $y\geq 4x$

II —"— : $y\geq -4x$

III kvadrant : $-y\geq -4x \Rightarrow y\leq 4x$

IV —"— : $-y\geq 4x \Rightarrow y\leq -4x$

zbog ln : $(x+y)^2-5(x+y)+6>0$; $x+y=:t$

$t^2-5t+6>0$, nultocke $t_1=2$, $t_2=3$

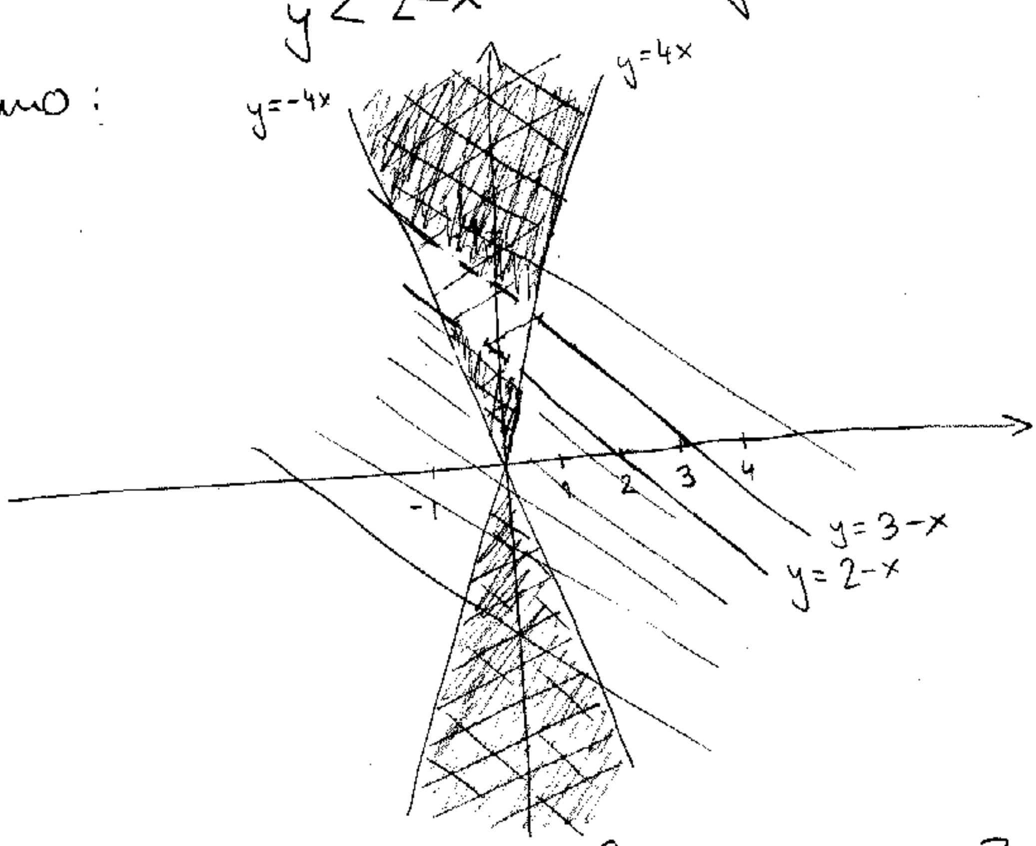
$\Rightarrow t\in(-\infty,2)\cup(3,+\infty)$

tj. $x+y\in(-\infty,2)\cup(3,+\infty)$

tj. $x+y<2$ ili $x+y>3$

$y<2-x$ ili $y>3-x$

Skiciramo:



Rješenje je osjenčano područje bez pravaca $y=3-x$, $y=2-x$.

(2) Izračunati $\frac{\partial^2 z}{\partial y \partial x}(0,0)$ ako je $z = \frac{x+y}{x+yz+2}$.

Rj:
$$zx + yz^2 + 2z - x - y = 0$$
$$F(x,y,z) = zx + yz^2 + 2z - x - y$$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{F_x}{F_z}(x,y,z) = -\frac{z-1}{x+2yz+2} = \frac{1-z}{x+2yz+2}$$

$$\frac{\partial^2 z}{\partial y \partial x}(x,y) = \frac{-\frac{\partial z}{\partial y}(x,y)(x+2yz+2) - (1-z)(2z+2y\frac{\partial z}{\partial y})}{(x+2yz+2)^2}$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{F_y}{F_z}(x,y,z) = -\frac{z^2-1}{x+2yz+2} = \frac{1-z^2}{x+2yz+2}$$

$z(0,0)$: $\quad z(0,0)\cdot 0 + 0\cdot z^2(0,0) + 2z(0,0) - 0 - 0 = 0$
$$\Rightarrow 2z(0,0) = 0$$
$$\Rightarrow z(0,0) = 0$$

$$\frac{\partial z}{\partial y}(0,0) = \frac{1 - z^2(0,0)}{0 + 2\cdot 0\cdot z(0,0) + 2} = \frac{1-0}{2} = \frac{1}{2}$$

$$\frac{\partial^2 z}{\partial y \partial x}(0,0) = \frac{-\frac{\partial z}{\partial y}(0,0)(0 + 2y\,z(0,0) + 2) - (1 - z(0,0))(2z(0,0) + 2\cdot 0\,\frac{\partial z}{\partial y}(0,0))}{(0 + 2\cdot 0\cdot z(0,0) + 2)^2} = \frac{-\frac{1}{2}\cdot 2 - (1-0)(2\cdot 0 + 0)}{2^2} = -\frac{1}{4}$$

(3) Na plohi $z = x^2 + y^2$ nađite točku najbližu točki $A(2,4,6)$.

Rj
$$d(T,A) = \sqrt{(x-2)^2 + (y-4)^2 + (z-6)^2} \quad \text{ako}$$
$$T(x,y,z)$$

jer je T na plohi: $z = x^2 + y^2 \Rightarrow$

$$d(T, A) = \sqrt{(x-2)^2+(y-4)^2+(x^2+y^2-6)^2}$$

$$\frac{\partial d}{\partial x}(x,y) = \frac{1}{2} \cdot \frac{1}{\sqrt{(x-2)^2+(y-4)^2+(x^2+y^2-6)^2}} \cdot \Big(2(x-2) +$$

$$+ 2(x^2+y^2-6)\cdot 2x\Big)$$

Da bi $\frac{\partial d}{\partial x}(x,y)$ bilo nula $\Rightarrow$

$$2(x-2)+2(x^2+y^2-6)\cdot 2x = 0$$
$$x-2+2x(x^2+y^2-6) = 0$$

$$\frac{\partial d}{\partial y}(x,y) = \frac{1}{2} \cdot \frac{1}{\sqrt{}} \cdot \big(2(y-4)+2(x^2+y^2-6)\cdot 2y\big)$$

$$\Rightarrow \quad y-4+2y(x^2+y^2-6) = 0 \Rightarrow y \neq 0$$
$$x-2+2x(x^2+y^2-6) = 0 \Rightarrow x \neq 0$$

$$\Rightarrow \quad \frac{y-4}{-2y} = \frac{x-2}{-2x} \Rightarrow \frac{y-4}{y} = \frac{x-2}{x} \Rightarrow -\frac{4}{y} = -\frac{2}{x}$$

i.e. $y = 2x$ i to uvrstimo nazad u gornju jednadžbu $\Rightarrow x-2+2x(x^2+(2x)^2-6)=0$

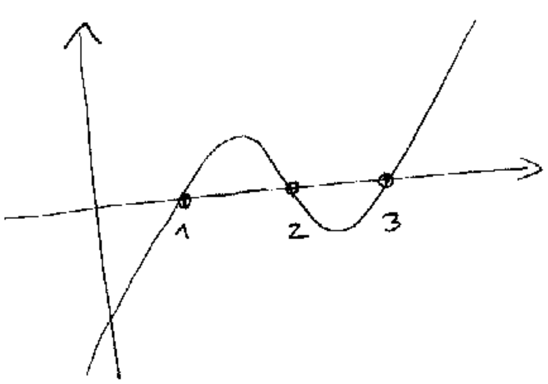
Dobivamo $10x^3-11x-2=0$ što nema lijepa rješenja i sva tri su realna.

Sad bi trebalo provjeriti drugu derivaciju s tim rješenjima, ali to ne treba zbog nerješivosti jedn. bez upotrebe računala.

(4) Izračunati površinu omeđenu sa $y=(x-1)(x-2)(x-$ i osi Ox.

Rj.

slika:



$$P = \int_1^2 (x-1)(x-2)(x-3)\,dx - \int_2^3 (x-1)(x-2)(x-3)\,dx =$$

jer je ovaj drugi integral negativnog predznaka

$$\int_1^2 (x^3-6x^2+11x-6)\,dx - \int_2^3 (x^3-6x^2+11x-6)\,dx =$$

$$\left.\begin{aligned}(x-1)(x-2)(x-3) &= (x-1)(x^2-5x+6) = x^3-\underline{5x^2}+6x-\underline{x^2}+5x-6 \\ &= x^3-6x^2+11x-6\end{aligned}\right\}$$

$$= \left(\frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x \Big|_1^2\right) - \left(\frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x \Big|_2^3\right) = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

5) Razviti u Taylorov red oko nule i odrediti domenu od $f(x) = \dfrac{1}{x^2-5x+4}$.

R:
$$x_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2}; \quad x_1 = 4, \; x_2 = 1$$

$$\Rightarrow \quad x^2-5x+4 = (x-4)(x-1)$$

$$\frac{1}{x^2-5x+4} = \frac{A}{x-4} + \frac{B}{x-1} \Rightarrow 1 = Ax - A + Bx - 4B$$

$$\Rightarrow \begin{aligned} A+B&=0 \quad B=-A \\ -A-4B&=1 \Rightarrow -A+4A=1 \Rightarrow 3A=1 \end{aligned}$$

$$A = \frac{1}{3}, \; B = -\frac{1}{3}$$

$$\frac{1}{x^2-5x+4} = \frac{1}{3}\cdot\frac{1}{x-4} - \frac{1}{3}\frac{1}{x-1} = -\frac{1}{12}\cdot\frac{1}{1-\frac{x}{4}} + \frac{1}{3}\cdot\frac{1}{1-x} =$$

$$= -\frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty} x^n =$$

$$= \sum_{n=0}^{\infty} -\frac{1}{12}\cdot\frac{x^n}{16^n} + \frac{1}{3}x^n = \sum_{n=0}^{\infty}\left(-\frac{1}{12\cdot 4^n} + \frac{1}{3}\right)x^n$$

i taj red konvergira tamo gdje oba gornja konv. tj. za $\left|\frac{x}{4}\right| < 1$ i $|x| < 1$ tj. za $|x| < 1$

i to je domena.

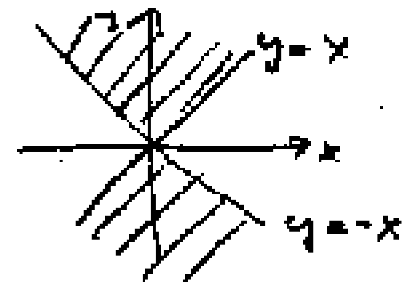
MATEMATIKA 2 – 23.9.2004 – RJEŠENJE

Zadatak. Odredite domenu funkcije $f(x,y)=\sqrt{y^2-x^2}+\ln(x^2+y^2-4)$ i skicirajte je u koordinatnoj ravnini.

Rješenje.

- $y^2-x^2\geq 0$ (zbog $\sqrt{\ }$)

$\Rightarrow y^2\geq x^2 \Rightarrow |y|\geq|x| \Rightarrow$ $\begin{matrix} y\geq x \\ y\geq -x\end{matrix}$ ili $\begin{matrix} y\leq x \\ y\leq -x\end{matrix}$

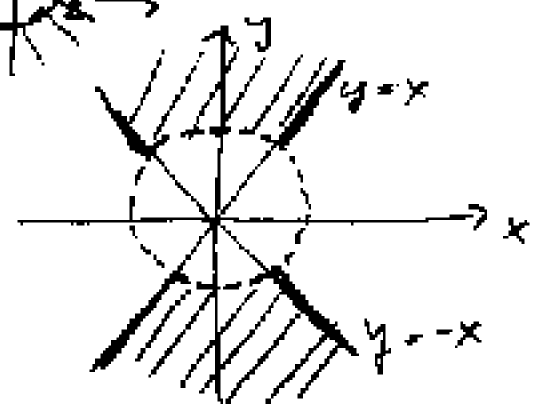


- $x^2+y^2-4>0$ (zbog ln)

$x^2+y^2>2^2$

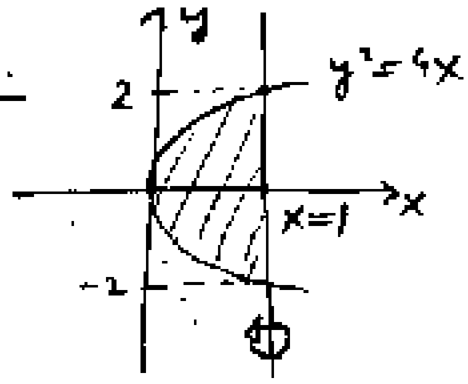
$S(0,0),\ r=2$

Konačno rješenje:

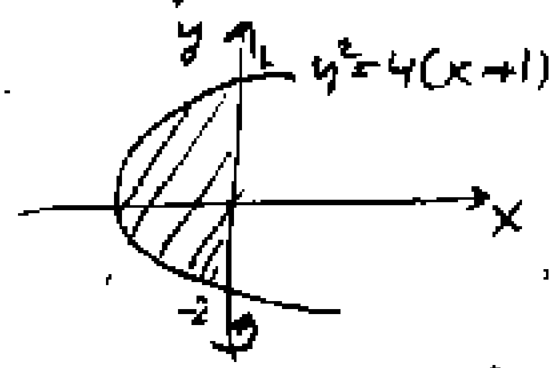


Zadatak. Nađite volumen tijela nastalog rotacijom oko pravca $x=1$ onog dijela parabole $y^2=4x$ koji odsijeca taj pravac.

Rješenje.



Naći ovaj volumen je isto kao naći volumen tijela nastalog rotacijom oko pravca $x=0$ onog dijela parabole $y^2=4(x+1)$ koji odsijeca $x=0$ (y-os).



$y^2=4x+4 \rightarrow x=g(y)=\frac{y^2}{4}-1$

$$\Rightarrow V=\pi\int_{-2}^{2}\left(\frac{y^2}{4}-1\right)^2dy=\pi\int_{-2}^{2}\left(\frac{y^4}{16}-\frac{1}{2}y^2+1\right)dy=$$

$$=\left(\frac{y^5}{80}-\frac{1}{2}\cdot\frac{1}{3}y^3+y\right)\Big|_{-2}^{2}=\dots=\boxed{\frac{32}{15}}$$

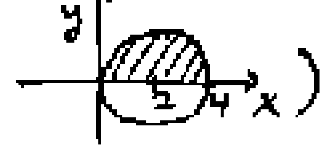
Zadatak. Prelaskom na polarne koordinate izračunajte

$$\iint_S (x^2+y^2)\,dx\,dy$$

gdje je S gornja polovica područja omeđenog kružnicom $x^2+y^2=4x$.

Rješenje. $\left.\begin{matrix} x=r\cos\varphi \\ y=r\sin\varphi\end{matrix}\right\}\Rightarrow r^2=4r\cos\varphi \Rightarrow r=4\cos\varphi$

$\varphi\in[0,\frac{\pi}{2}]$ (gornja polovica kružnice $x^2+y^2=4x$, tj. $S(2,0),\ r=2$:



)

$\Rightarrow$ integral glasi:

$$\int_0^{\pi/2}\int_0^{4\cos\varphi} r^2\cdot r\,dr\,d\varphi=$$

$$= \int_0^{\pi/2}\left(\frac{r^4}{4}\right)\Big|_0^{4\cos\varphi} d\varphi = 64\int_0^{\pi/2}(\cos^4\varphi)\,d\varphi = 64\int_0^{\pi/2}(\cos^2\varphi)^2\,d\varphi =$$

$$= 64\int_0^{\pi/2}\left(\frac{1+\cos2\varphi}{2}\right)^2 d\varphi = 16\int_0^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)\,d\varphi =$$

$$= 16\int_0^{\pi/2}\left(1+2\cos2\varphi+\frac{1+\cos4\varphi}{2}\right)d\varphi = 16\cdot(\varphi)\Big|_0^{\pi/2} + 32\cdot\left(\frac{\sin2\varphi}{2}\right)\Big|_0^{\pi/2} +$$

$$+ 8(\varphi)\Big|_0^{\pi/2} + 8\left(\frac{\sin4\varphi}{4}\right)\Big|_0^{\pi/2} = 8\pi + 16\cdot\left(\frac{\sin\pi}{2} - \frac{\sin0}{2}\right) + \ldots$$

$$4\pi + 8\left(\frac{\sin2\pi}{4} - \frac{\sin0}{4}\right) = 12\pi$$

Zadatak. Nađite ekstreme funkcije $z(x,y)$ zadane implicitno s
$$x^3 - y^2 - 3x + 4y + z^2 + z - 8 = 0.$$

Rješenje. $F(x,y,z) = z^2+z-8+x^3-y^2-3x+4y$

$F_x = 3x^2-3,\quad F_y = -2y+4,\quad F_z = 2z+1$

$$\Rightarrow\quad z_x = -\frac{3x^2-3}{2z+1},\quad z_y = -\frac{-2y+4}{2z+1}$$

$$z_x = z_y = 0 \Rightarrow x=\pm1,\ y=2$$

$\Rightarrow$ točke-kandidati za ekstrem su $(1,2)$ i $(-1,2)$.

$\boxed{(1,2)}$:

$$z_{xx} = -\frac{6x(2z+1)-(3x^2-3)\cdot 2z_x}{(2z+1)^2} = -\frac{6x}{2z+1}$$

$$z_{yx} = -\frac{0\cdot(2z+1)-(3x^2-3)\cdot 2z_y}{(2z+1)^2} = 0 = z_{xy}$$

$$z_{yy} = -\frac{-2\cdot(2z+1)-(-2y+4)\cdot 2z_y}{(2z+1)^2} = \frac{2}{2z+1}.$$

$z(1,2)=?$ uvrstimo u početnu jednakost $x=1, y=2$:

$$1-4-3+\cancel{8}+z^2+z-\cancel{8}=$$
$$z^2+z-6=0$$
$$z_{1,2} = \frac{-1\pm\sqrt{1+24}}{2} = \frac{-1\pm5}{2}$$
$$z_1=-3,\ z_2=2$$

$(1,2,-3)$: $z_{xx} = -\frac{6}{-5} = \frac{6}{5}$, $z_{yy} = \frac{2}{-6+1} = -\frac{2}{5}$

$\Rightarrow \Delta = \frac{6}{5}\cdot\frac{-2}{5} = \frac{-12}{25} < 0 \Rightarrow$ nema ekstrema

$(1,2,2)$: $z_{xx} = \frac{-6}{5}$, $z_{yy} = \frac{2}{5} \Rightarrow \Delta = -\frac{12}{25} < 0 \Rightarrow$ nema ekstrema

$\boxed{(-1,2)}$: $z(-1,2)=?$ $\quad -1-4+3+\cancel{8}+z^2+z-\cancel{8}=0$

$$z^2+z-2=0$$
$$z_{1,2} = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm3}{2} \Rightarrow z_1=-2,\ z_2=1$$

$(-1,2,-2)$: $z_{xx} = \frac{6}{-3} = -2$, $z_{yy} = -\frac{2}{3}$

$\Rightarrow \Delta>0,\ z_{xx}<-2 \Rightarrow$ lok. maksimum

$(-1,2,1)$: $z_{xx} = \frac{6}{3} = 2$, $z_{yy} = \frac{2}{3}$

$\Delta > 0$, $z_{xx} = 2 > 0 \Rightarrow$ tu je lok. minimum

<u>Zaključak</u>: u $((-1,2),-2)$ postiže se lok. maksimum, a u $((-1,2),1)$ lokalni minimum

---

<u>Zadatak</u>. Razvijte u Taylorov red oko $x_0 = 0$ funkciju $f(x) = \cos 2x + \frac{1}{x-2}$. Odredite područje konvergencije tog reda i $f^{(100)}(0)$.

<u>Rješenje</u>:

Taylorov red od $\cos x$: $\cos x = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\Rightarrow \cos 2x = \sum_{n\geq 0} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n\geq 0} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

Funkcija $f_2(x) = \frac{1}{x-2}$ se razvija u red pomoću geom. reda:

$$\frac{1}{x-2} = -\frac{1}{2-x} = -\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$$

$$\Rightarrow \cos 2x + \frac{1}{x-2} = \sum_{n\geq 0} (-1)^n \frac{2^{2n}x^{2n}}{(2n)!} - \frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$$

$\cos 2x$ konv. za svako $x \in \mathbb{R}$, a $\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$ za $\left|\frac{x}{2}\right| < 1$

$\Rightarrow$ područje konvergencije je $\left|\frac{x}{2}\right| < 1$ tj. $|x| < 2$.

$\frac{f^{(100)}(0)}{100!}$ = koef. koji se nalazi uz x^{100} u oba reda

$$\Rightarrow \frac{f^{(100)}(0)}{100!} = (-1)^{50}\frac{2^{100}}{(100)!} - \frac{1}{2}\frac{1}{2^{100}} =$$

$$= \frac{2^{100}}{(100)!} - \frac{1}{2^{101}}$$

$$\Rightarrow f^{(100)}(0) = 2^{100} - \frac{(100)!}{2^{101}}$$

MAT II - 1.10.2004. Rješenja

1. Odredite domenu od $f(x,y)=\sqrt{\sin(x+y)}-\sqrt{16\pi^2-(x+y)^2}$

Rj.

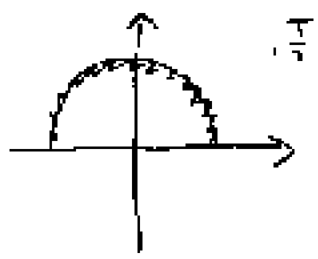
zbog $\sqrt{\ }$: $\sin(x+y)\geq 0$

$16\pi^2-(x+y)^2\geq 0 \Rightarrow 16\pi^2\geq (x+y)^2 \ /\sqrt{\ }$

$4\pi \geq |x+y|$

$\Rightarrow -4\pi \leq x+y \leq 4\pi$

vraćamo se na $\sin(x+y)\geq 0 \Rightarrow$



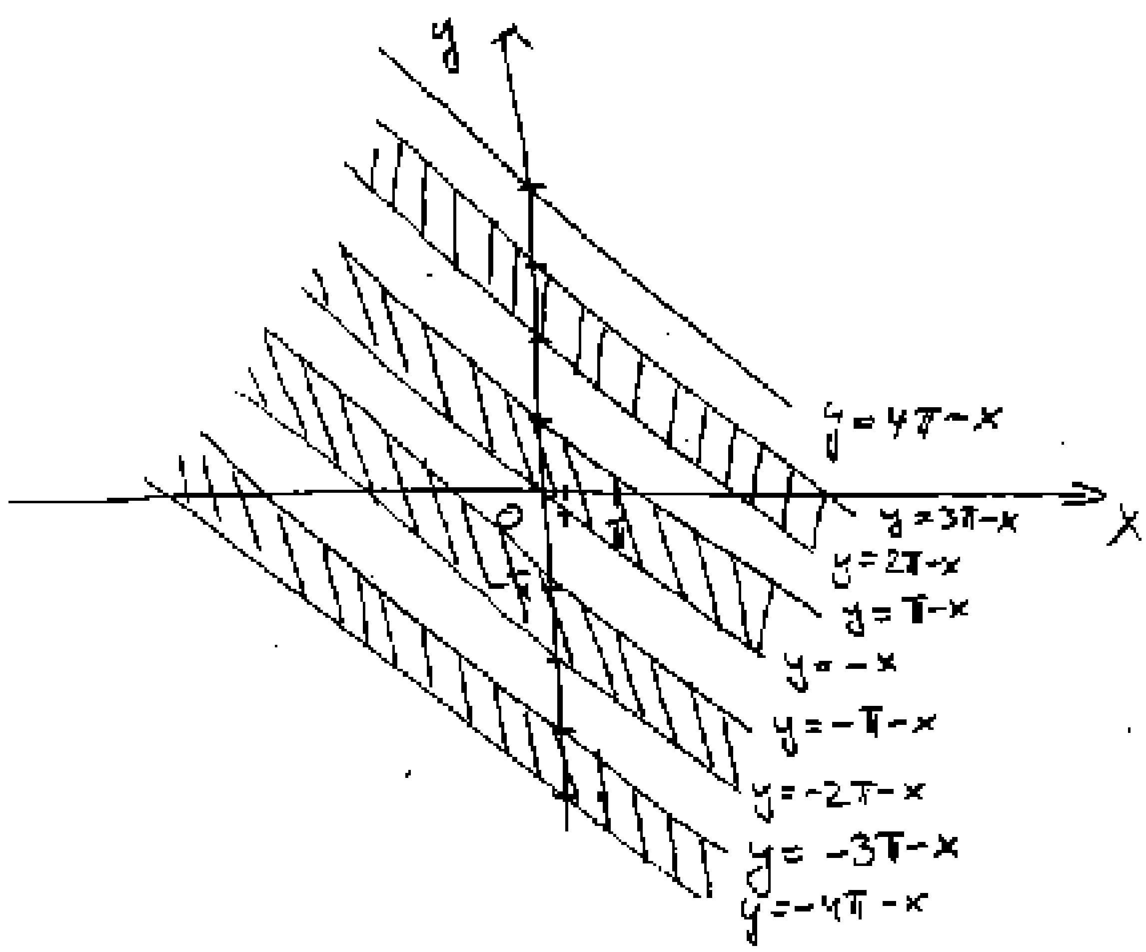
$x+y \in \ldots[-2\pi,-\pi]\cup[0,\pi]\cup[2\pi,3\pi]\cup\ldots$

zbog $-4\pi\leq x+y\leq 4\pi$ je $x+y\in[-4\pi,-3\pi]\cup[-2\pi,-\pi]\cup[0,\pi]\cup$
$\cup[2\pi,3\pi]\cup\{4\pi\}$

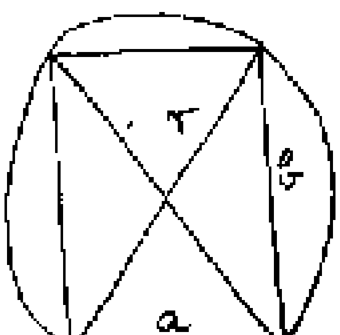
$x+y\geq -4\pi \quad y\geq -4\pi-x$

$x+y\leq -3\pi \quad y\leq -3\pi-x \quad$ itd. ostale

Skica:



2. Izračunajte približno polumjer opisane kružnice pravokutnika stranica $a=5.1$ i $b=11.8$



$$r=\frac{d}{2}=\sqrt{a^2+b^2}$$

$$r(a,b)=r(a_0,b_0)+\frac{\partial r}{\partial a}(a_0,b_0)\Delta a+$$
$$+\frac{\partial r}{\partial b}(a_0,b_0)\Delta b$$

$a_0=5, \ \Delta a+a_0=5.1 \Rightarrow \Delta a=0.1$

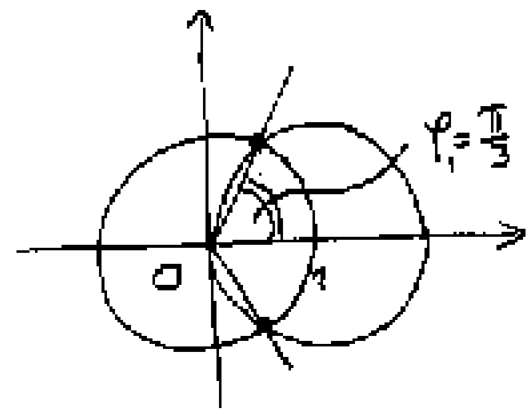
$b_0 = 12$, $b_0 + \Delta b = 11.8 \Rightarrow \Delta b = -0.2$

$$r(5.1, 11.8) = \sqrt{a_0^2+b_0^2} + \frac{a_0}{\sqrt{a_0^2+b_0^2}}\Delta a + \frac{b_0}{\sqrt{a_0^2+b_0^2}}\Delta b =$$

$$= 13 + \frac{5}{13}\cdot\frac{1}{10} - \frac{12}{13}\cdot\frac{2}{10} = 13 + \frac{5}{130} - \frac{24}{130} = 13 - \frac{19}{130}$$

3. Prelaskom na polarne koordinate izračunajte dvostruki integral $\iint\limits_{(S)} \sqrt{x^2+y^2}\,dx\,dy$ gdje je S područje omeđeno kružnicama $x^2+y^2=1$ i $(x-1)^2+y^2=1$

Rj



$$(x-1)^2+y^2=1$$
$$x^2-2x+\cancel{1}+y^2=\cancel{1}$$
$$x^2-2x+y^2=0$$
$$r^2\cos^2\varphi - 2r\cos\varphi + r^2\sin^2\varphi = 0$$
$$r_1 = 2\cos\varphi$$
$$x^2+y^2=1 \Rightarrow r_2 = 1$$

Tražimo sjecišta kružnica.
Lagano se vidi da $x = 1/2$, $y^2 = \frac{3}{4}$ $y_1 = \frac{\sqrt{3}}{2}$, $y_2 = -\frac{\sqrt{3}}{2}$

$$\operatorname{tg}\varphi_1 = \frac{y_1}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \varphi_1 = \frac{\pi}{3}, \text{ analogno } \varphi_2 = -\frac{\pi}{3}$$

$$\iint\limits_{(S)}\sqrt{x^2+y^2}\,dx\,dy = \int_{-\frac{\pi}{2}}^{-\pi/3} d\varphi \int_0^{2\cos\varphi} r\cdot r\,dr + \int_{-\pi/3}^{\pi/3} d\varphi \int_0^{1} r\cdot r\,dr + \int_{\pi/3}^{\pi/2} d\varphi \int_0^{2\cos\varphi} r\cdot r\,dr =$$

$$= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \frac{1}{3}r^3\Big|_0^{2\cos\varphi} d\varphi + \int_{-\pi/3}^{\pi/3} \frac{1}{3}r^3\Big|_0^1 d\varphi + \int_{\pi/3}^{\pi/2} \frac{1}{3}r^3\Big|_0^{2\cos\varphi} d\varphi =$$

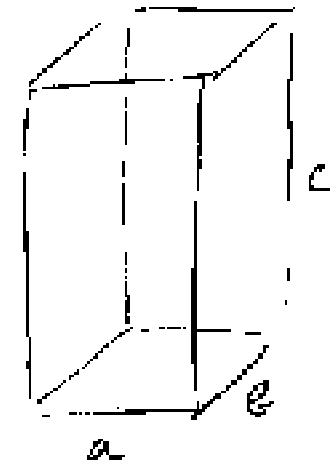
$$= \int_{-\pi/2}^{-\pi/3} \frac{8}{3}\cos^3\varphi\,d\varphi + \frac{1}{3}\int_{-\frac{\pi}{3}}^{\pi/3} d\varphi + \int_{\pi/3}^{\pi/2} \frac{8}{3}\cos^3\varphi\,d\varphi =$$

$$= -\frac{8}{3}\cdot\frac{1}{4}(1-\sin^2\varphi)^2\Big|_{-\pi/2}^{-\pi/3} + \frac{1}{3}\cdot\frac{2\pi}{3} - \frac{8}{3}\cdot\frac{1}{4}(1-\sin^2\varphi)^2\Big|_{\pi/3}^{\pi/2} =$$

$$= -\frac{2}{3}\left[(1-\frac{3}{4})^2 - 1\right] + \frac{2\pi}{9} - \frac{2}{3}\left[(1-0)^2 - (1-\frac{3}{4})^2\right] =$$

$$= -\frac{2}{3}\cdot\left(-\frac{15}{16}\right) + \frac{2\pi}{9} - \frac{2}{3}\cdot\frac{15}{16} = \frac{2\pi}{9}$$

4. Među svim kvadrima oplošja 2 odredite onaj najvećeg obujma.



Rj

$$V = abc$$

$$O_p = 2ab + 2bc + 2ac$$

$$\cancel{2} = \cancel{2}ab + \cancel{2}bc + \cancel{2}ac$$

$$1 = ab + c(b+a)$$

$$c = \frac{1-ab}{b+a}$$

$$V = ab\,\frac{1-ab}{b+a} = \frac{ab-(ab)^2}{b+a}$$

$$\frac{\partial V}{\partial a}(a,b) = \frac{(b-2ab\cdot b)(b+a)-(ab-(ab)^2)}{(b+a)^2} = 0$$

$$\frac{\partial V}{\partial b}(a,b) = \frac{(a-2ba\cdot a)(b+a)-(ab-(ab)^2)}{(b+a)^2} = 0$$

$$b(1-2ab)(b+a) - ab(1-ab) = 0 \qquad b \neq 0$$

$$a(1-2ab)(b+a) - ab(1-ab) = 0 \qquad a \neq 0$$

$$\Rightarrow \left.\begin{array}{l}(1-2ab)(b+a) - a(1-ab) = 0\\ (1-2ab)(b+a) - b(1-ab) = 0\end{array}\right\} \; a = b \quad \text{ili} \quad 1 = ab$$

$$(1-2a^2)2a - a(1-a^2) = 0$$

$$2(1-2a^2) - (1-a^2) = 0$$

$$2 - 4a^2 - 1 + a^2 = 0$$

$$-3a^2 = -1$$

$$a^2 = \frac{1}{3} \qquad a = \frac{\sqrt{3}}{3}, \quad b = \frac{\sqrt{3}}{3}$$

$ab = 1$

$$-(b+a) = 0$$

$$b = -a$$

$$\Rightarrow -a(1+a^2) = 0 \Rightarrow a = 0 \quad \text{ili} \quad a^2 = -1$$

Sada se još moraju ispitati druge derivacije u točki $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

5. Razvijte u Taylorov red oko $x_0 = 1$ funkciju

$$f(x) = \frac{1}{x} + \frac{1}{3-x}$$

i ispitajte područje konvergencije tog reda.

Rj

$$\frac{1}{x} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \text{ konv. za } |x-1| < 1$$

$$\frac{1}{3-x} = \frac{1}{2-(x-1)} = \frac{1}{2} \frac{1}{1-\frac{x-1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}, \text{ konv. za } \left|\frac{x-1}{2}\right| < 1$$

sve skupa:

$$f(x) = \frac{1}{x} + \frac{1}{3-x} = \sum_{n=0}^{\infty} (x-1)^n (-1)^n + \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}}\right) (x-1)^n, \text{ konv } |1-x| < 1$$

tj. $x \in (0,2)$.

MAT II - 13/11/04 - RJEŠENJA

① Naći $\mathcal{D}(f)$ ako je $f(x,y) = \sqrt{(x+y)\cos(xy)}$

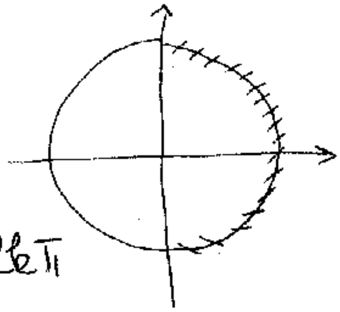
Rj: zbog $\sqrt{}$: $(x+y)\cos(xy) \geq 0$

a) $x+y \geq 0$, $\cos(xy) \geq 0$

$y \geq -x$

Gledamo uvjet $\cos(xy) \geq 0$

$\Rightarrow$ (vidi sličicu) $xy \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] + 2k\pi$

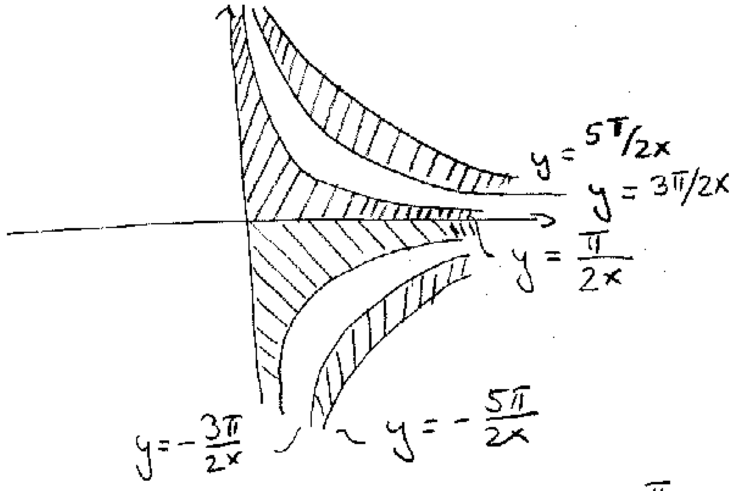


$\Rightarrow xy \in \ldots \cup \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \cup \ldots$

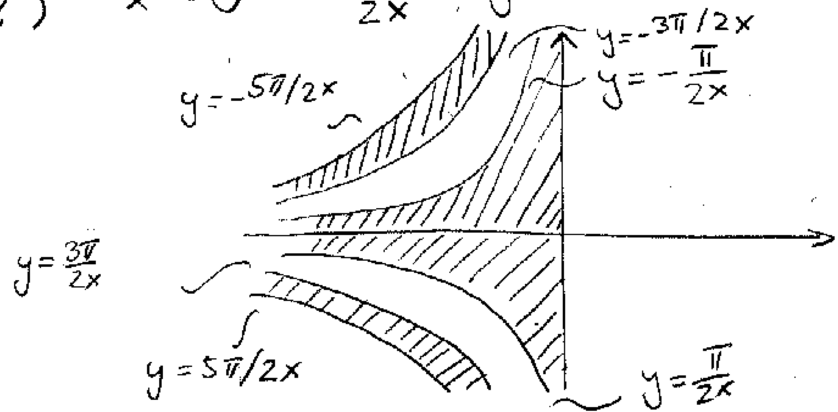
Promotrimo svaki taj interval i crtamo; npr. prva tri: $xy \in \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$

$\Rightarrow -\frac{5\pi}{2} \leq xy \leq -\frac{3\pi}{2}, \quad -\frac{\pi}{2} \leq xy \leq \frac{\pi}{2}, \quad \frac{3\pi}{2} \leq xy \leq \frac{5\pi}{2}$

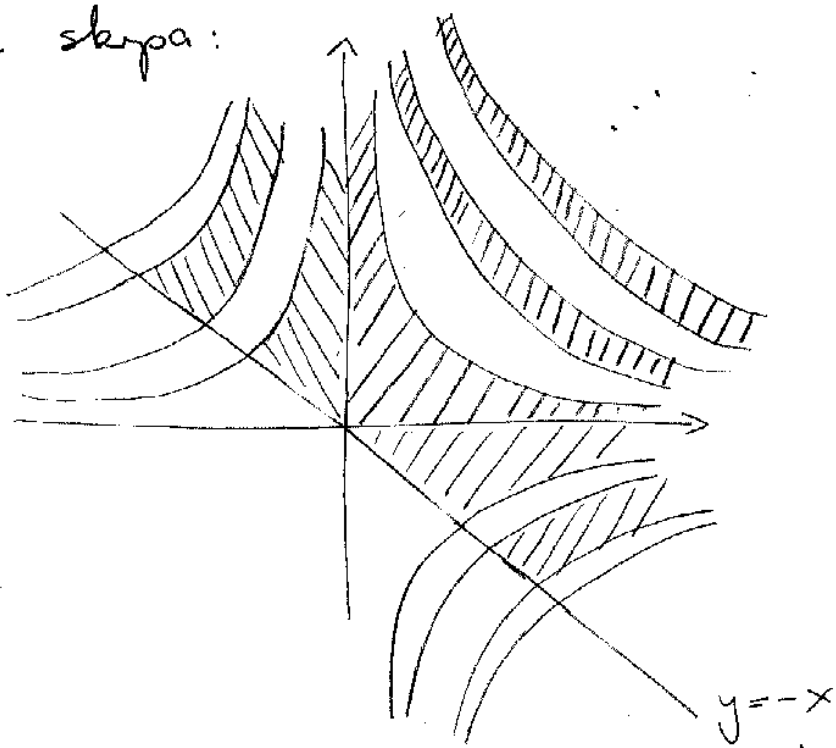
1) $x > 0 \Rightarrow -\frac{5\pi}{2x} \leq y \leq -\frac{3\pi}{2x}, \quad -\frac{\pi}{2x} \leq y \leq \frac{\pi}{2x}, \quad \frac{3\pi}{2x} \leq y \leq \frac{5\pi}{2x}$



2) $x < 0 \quad -\frac{3\pi}{2x} \leq y \leq -\frac{5\pi}{2x}, \quad \frac{\pi}{2x} \leq y \leq -\frac{\pi}{2x}, \quad \frac{5\pi}{2x} \leq y \leq \frac{3\pi}{2x}$



pa sve skupa:



) $x+y \le 0$ $\cos(xy) \le 0$ analogno teko.

načno nj. a) ∪ b).

) sfera je zadana jednadžbom $x^2+y^2+z^2-1=0$. Odredite jednadžbu tang. ravnine koja sadrži točke $T_1(1,3,1)$ i $T_2(2,1,1)$.

$F_x(x_0,y_0,z_0)(X-x_0)+F_y(x_0,y_0,z_0)(Y-Y_0)+F_z(x_0,y_0,z_0)(Z-z_0)=0$

je jdn. tang. ravnine u $T_0(x_0,y_0,z_0)$

$\Rightarrow 2x_0(X-x_0)+2y_0(Y-Y_0)+2z_0(Z-z_0)=0$

Jer T_1 i T_2 leže u toj ravnini:

$2x_0(2-x_0)+2y_0(1-y_0)+2z_0(1-z_0)=0$

$2x_0(1-x_0)+2y_0(3-y_0)+2z_0(1-z_0)=0$

$$\left.\begin{aligned}2x_0-x_0^2+y_0-y_0^2+z_0-z_0^2=0\\ x_0-x_0^2+3y_0-y_0^2+z_0-z_0^2=0\end{aligned}\right\} -$$

$x_0^2+y_0^2+z_0^2=1$ $\quad 2x_0+y_0+z_0-1=0$

$x_0+3y_0+z_0-1=0$

$\Rightarrow 2x_0+y_0=x_0+3y_0$

$\Rightarrow x_0=2y_0$

$\Rightarrow 4y_0+y_0+z_0-1=0$

$z_0=1-5y_0$

$\Rightarrow (2y_0)^2 + y_0^2 + (1-5y_0)^2 - 1 = 0$

$4y_0^2 + y_0^2 + \cancel{1} - 10y_0 + 25y_0^2 - \cancel{1} = 0$

$30y_0^2 - 10y_0 = 0 \Rightarrow 3y_0^2 - y_0 = 0$

i) $y_0 = 0$ ii) $y_0 = \frac{1}{3}$

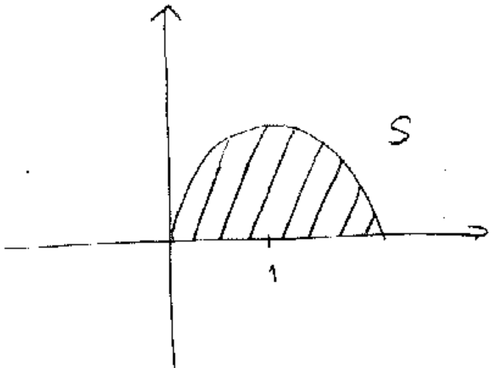
$x_0 = 0$ $x_0 = \frac{2}{3}$

$z_0 = 1$ $z_0 = -\frac{2}{3}$

i rješenja su ravnine u tim točkama.

) Izračunati $\iint\limits_{(S)} \sqrt{x^2+y^2}\, dx\, dy$ gdje je S područje omeđeno sa $(x-1)^2 + y^2 = 1$ i osi OX.

Rj



Prelazimo na polarne koordinate: $r^2\cos^2\varphi - 2r\cos\varphi + r^2\sin^2\varphi = 0$

$r^2 = 2r\cos\varphi$

$r = 2\cos\varphi$

$$\iint\limits_{(S)} \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \sqrt{r^2} \cdot r\, dr =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\, dr = \int_0^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \Big|_0^{2\cos\varphi} \right) d\varphi$$

$$= \frac{1}{3} \int_0^{\pi/2} 8\cos^3\varphi\, d\varphi = \frac{8}{3} \int_0^{\frac{\pi}{2}} (1-\sin^2\varphi)\cos\varphi\, d\varphi =$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi - \frac{8}{3} \int_0^{\frac{\pi}{2}} \sin^2\varphi \cos\varphi\, d\varphi = \frac{8}{3}(0-1) -$$

$$- \frac{8}{3} \left(\frac{1}{3} \sin^3\varphi \Big|_0^{\frac{\pi}{2}} \right) = -\frac{8}{3} - \frac{8}{3} \cdot \frac{1}{3} = -\frac{8}{3} - \frac{8}{9}$$

④ Od svih trokuta zadanog opsega 2 nađite onaj koji ima najveću površinu.

Rj. $O = a+b+c = 2$

$P = \sqrt{s(s-a)(s-b)(s-c)}$ $\quad s = \frac{a+b+c}{2} = \frac{O}{2} = 1$

$P = \sqrt{(1-a)(1-b)(1-c)}$ $\quad c = 2-a-b$

$1-c = 1-(2-a-b) = a+b-1$

$P = \sqrt{(1-a)(1-b)(a+b-1)} =$

$= \sqrt{(1-a-b+ab)(a+b-1)} = \sqrt{\cancel{a}+\cancel{b}-1-a^2-\cancel{ab}+\cancel{a}-\cancel{ab}-b^2+\cancel{b}+a^2b+ab^2-\cancel{ab}}$

$= \sqrt{2a+2b-1-3ab+a^2b+ab^2-a^2-b^2}$

$$\left.\frac{\partial P}{\partial a}\right|_{(a,b)} = \frac{2-3b+2ab+b^2-2a}{\sqrt{}}\cdot\frac{1}{2} = 0$$

$$\left.\frac{\partial P}{\partial b}\right|_{(a,b)} = \frac{2-3a+a^2+2ab-2b}{\sqrt{}}\cdot\frac{1}{2} = 0$$

$\Rightarrow$

$\Rightarrow$ $\left.\begin{array}{l}2-3b+2ab+b^2-2a=0\\ 2-3a+2ab+a^2-2b=0\end{array}\right\}-$ $/ \ -3b+3a+b^2-a^2-2a+2b$

$\Rightarrow a-b+b^2-a^2=0$

$\Rightarrow a-b-(a-b)(a+b)=0 \Rightarrow (a-b)(1-(a+b))=$

$\Rightarrow a=b$

ili $1-a-b=0$

$a = 1-b$

i) $a=b$

$2-3a+2a^2+a^2-2a=0$

$3a^2-5a+2=0 \quad 3a^2-3a-2a+2=0$

$3a(a-1)-2(a-1)=0 \Rightarrow a=1$

$a=\frac{2}{3}$

$a=1 \Rightarrow b=1 \Rightarrow c=0$ ↯

$a=\frac{2}{3} \Rightarrow b=\frac{2}{3} \Rightarrow c=2-\frac{4}{3}=\frac{2}{3}$ ✓

ii) $2-3b+2(1-b)b+b^2-2(1-b)=0$

$\cancel{2}-3b+2b-2b^2+b^2-\cancel{2}+2b=0$

$-b^2+b=0 \Rightarrow b=0$ ili $-b+1=0$ tj. $b=1$

$\Rightarrow a=0$ ↯

Jedina mogućnost $a=b=\frac{2}{3}$ i to se provjeri sa drugim derivacijama.

(5) $y''-2y'=e^{2x}+5$

Naći partikularno rješ. koje zadovoljava $y(0)=y'(0)=0$

Rj: $\lambda^2-2\lambda=0 \quad \lambda_1=0 \quad \lambda_2=2$

$y_0=C_1+C_2e^{2x}$ je homogeno rješenje.

$f_1(x)=e^{2x}$

Tražimo partikularno za f_1:

$$Y=xe^{2x}\cdot A$$

$$Y'=Ae^{2x}+2Axe^{2x}$$

$$Y''=2Ae^{2x}+2Ae^{2x}+4Axe^{2x}=4Ae^{2x}+4Axe^{2x}$$

$$4A\cancel{x}e^{2x}+4Ae^{2x}-2Ae^{2x}-4\cancel{Ax}e^{2x}=e^{2x}$$

$$2Ae^{2x}=e^{2x} \Rightarrow A=\frac{1}{2}$$

$$Y_1=\frac{1}{2}xe^{2x}$$

$f_2(x)=5$, opet isto, $Y=Bx$

$Y'=B$, $Y''=0$

$\Rightarrow 0-2B=5 \Rightarrow B=-\frac{5}{2} \Rightarrow Y_2=-\frac{5}{2}x$

$$Y_P=\frac{1}{2}xe^{2x}-\frac{5}{2}x$$

$$Y=C_1+C_2e^{2x}+\frac{1}{2}xe^{2x}-\frac{5}{2}x$$

$Y(0)=0 \Rightarrow C_1+C_2=0$

$Y'=2C_2e^{2x}+\frac{1}{2}e^{2x}+xe^{2x}-\frac{5}{2}$

$Y'(0)=0 \Rightarrow 2C_2+\frac{1}{2}-\frac{5}{2}=0 \quad 2C_2-2=0$

$C_2=1$

$C_1=-1$

$$Y=-1+e^{2x}+\frac{1}{2}xe^{2x}-\frac{5}{2}x$$

MAT II – 11.12.2004. RJEŠENJA

1. Odredite $\mathcal{D}(f)$ ako je

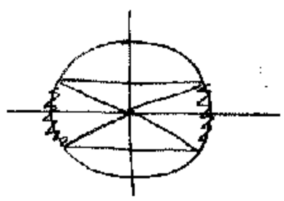
$$f(x,y) = \sqrt{\left(\tfrac{1}{4}-\sin^2 x\right)\left(\tfrac{1}{4}-\cos^2 y\right)} + \ln\left(2\pi - (x^2+y^2)\right)$$

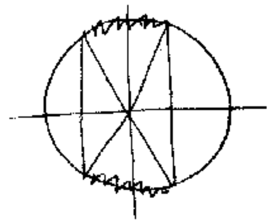
Rj. zbog ln: $x^2+y^2 < 2\pi$

zbog $\sqrt{}$: $\left(\frac{1}{4}-\sin^2 x\right)\left(\frac{1}{4}-\cos^2 y\right) \geq 0$

a) $\frac{1}{4} \geq \sin^2 x$, $\frac{1}{4} \geq \cos^2 y$

$\Rightarrow$ $|\sin x| \leq \frac{1}{2}$, $|\cos y| \leq \frac{1}{2}$

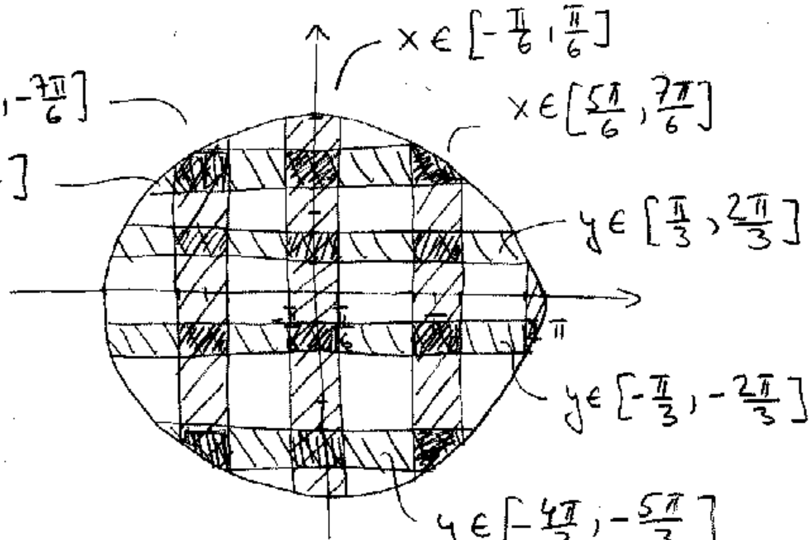




$$\Rightarrow x \in \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right] \cup \left[\frac{11\pi}{6}, \frac{13\pi}{6}\right] + 2k\pi$$

$$y \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] + 2k\pi$$

sve skupa



Analogno tako slučaj b), $\frac{1}{4} \leq \sin^2 x$, $\frac{1}{4} \leq \cos^2 x$.
Konačno rješenje: a) $\cup$ b).

2. Tang. ravnina na sferu $x^2+y^2+z^2-1=0$ koja sadrži točke $T_1(3,2,1)$ i $T_2(2,3,1)$

Rj.
$$2x_0(X-x_0) + 2y_0(Y-y_0) + 2z_0(Z-z_0) = 0$$

jdb. tang. ravnine u $T(x_0,y_0,z_0)$ na sferi.

$$\Rightarrow \quad 2x_0(3-x_0)+2y_0(2-y_0)+2z_0(1-z_0)=0$$
$$i \quad 2x_0(2-x_0)+2y_0(3-y_0)+2z_0(1-z_0)=0$$
$$\Rightarrow \quad 2x_0(3-\cancel{x_0}-2+\cancel{x_0})+2y_0(2-\cancel{y_0}-3+\cancel{y_0})=0$$
$$2x_0-2y_0=0 \Rightarrow x_0=y_0$$
$$\Rightarrow \quad 4x_0-2x_0^2+6x_0-2x_0^2+2z_0-2z_0^2=0$$
iz jdn. sfere i.. $x_0^2+x_0^2+z_0^2=1$
$$10x_0+2z_0-2(2x_0^2+z_0^2)=0$$
$$10x_0+2z_0-2=0$$
$$2z_0=2-10x_0$$
$$z_0=1-5x_0$$
$$\Rightarrow \quad 2x_0^2+(1-5x_0)^2=1$$
$$2x_0^2+\cancel{1}-10x_0+25x_0^2=\cancel{1} \Rightarrow 27x_0^2-10x_0=0$$

a) $x_0=0$, $y_0=0$, $z_0=1$

b) $x_0=\frac{10}{27}$, $y_0=\frac{10}{27}$, $z_0=1-\frac{50}{27}=-\frac{23}{27}$

. Naći lokalne ekstreme fkcije $x^2+y^2+z^2=1+2x+z$

Rj. $F(x,y,z)=x^2+y^2+z^2-2x-z-1$
$$\partial_x z=-\frac{\partial_x F}{\partial_z F}=-\frac{2x-2}{2z-1}, \quad \partial_y z=-\frac{\partial_y F}{\partial_z F}=-\frac{2y}{2z-1}$$
$\partial_x z=\partial_y z=0 \Rightarrow x=1, y=0$ i to je kandidat.

$$\cancel{1}+z^2(1,0)=\cancel{1}+2+z(1,0)$$
$$z^2-z-2=0$$
$$z=\frac{1\pm\sqrt{1+8}}{2}$$
$$z_1=2, \quad z_2=-1$$

$$\partial_{xx} z=-\frac{2(2z-1)-(2x-2)2\partial_x z}{(2z-1)^2}$$

gledamo prvi slučaj, $z_1(1,0)=2$
$$\partial_{xx} z=-\frac{2(2z-1)-(2x-2)2\cdot\frac{\partial z}{\partial x}}{(2z-1)^2} \Rightarrow$$

$$\partial_{xx}(1,0) = -\frac{2(\cancel{2\cdot 2-1})-0}{(2\cdot 2-1)^{\cancel{2}}} = -\frac{2}{3}$$

$$\partial_{xy} z = -\frac{0\cdot(2z-1)-(2x-2)\cdot 2\cdot \partial_y z}{(2z-1)^2} \Rightarrow \partial_{xy} z(1,0) = 0$$

$$\partial_{yy} z = -\frac{2(2z-1)-2y\cdot 2\cdot \partial_y z}{(2z-1)^2} \Rightarrow \partial_{yy} z(1,0) = -\frac{2}{(2z-1)} = -\frac{2}{3}$$

$$\Delta = \left(-\frac{2}{3}\right)^2 - 0 = \frac{4}{9} > 0 \Rightarrow$$ tu je lok. maximum jer $A < 0$

Gledamo drugi slučaj, $z_2 = -1$

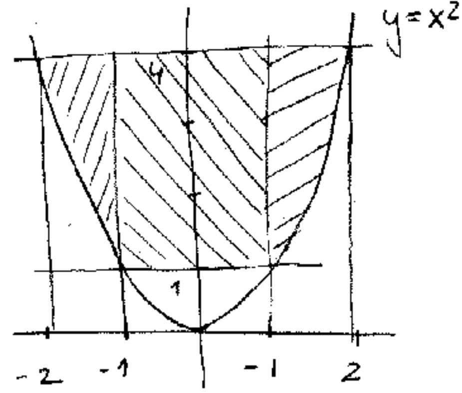
$$\partial_{xx} z(1,0) = -\frac{2}{(2\cdot(-1)-1)} = \frac{2}{3}$$

i analogno $\partial_{xy} z(1,0) = 0$, $\partial_{yy} z(1,0) = \frac{2}{3}$

$\Rightarrow$ $\Delta = \frac{4}{9}$ i za tu funkciju $(1,0)$ je lok. minimum jer $A > 0$.

4. Izračunati integral: $\int_{-2}^{-1} dx \int_{x^2}^{4} x^2 dy + \int_{-1}^{1} dx \int_{1}^{4} x^2 dy + \int_{1}^{2} dx \int_{x^2}^{4} x^2 dy$

Rj Crtamo područje integr.:



Promijenimo poredak integracije i I. postaje:

$$\int_1^4 dy \int_{-\sqrt{y}}^{\sqrt{y}} x^2 dx = \int_1^4 dy \left(\frac{1}{3}x^3 \Big|_{-\sqrt{y}}^{\sqrt{y}}\right) = \int_1^4 \frac{1}{3}\left(y\sqrt{y}+y\sqrt{y}\right) dy =$$

$$= \frac{2}{3}\int_1^4 y\sqrt{y}\, dy = \frac{2}{3}\cdot\frac{2}{5}\left(y^{\frac{5}{2}} \Big|_1^4\right) = \frac{4}{15}\left(2^5 - 1\right)$$

5. Razviti u Taylorov red $f(x)=\frac{x}{9-x^2}$ i naći domenu

Rj: $\frac{1}{9-x^2}=\frac{A}{3-x}+\frac{B}{3+x} \Rightarrow 1=3A+Ax+3B-Bx$

$A-B=0 \Rightarrow A=B$

$3A+3B=1$

$6A=1 \Rightarrow A=B=\frac{1}{6}$

$$f(x)=x\cdot\frac{1}{9-x^2}=x\left(\frac{1}{6}\cdot\frac{1}{3-x}+\frac{1}{6}\frac{1}{3+x}\right)=$$

$$=\frac{x}{6}\left(\frac{1}{3}\cdot\frac{1}{1-\frac{x}{3}}+\frac{1}{3}\frac{1}{1-(-\frac{x}{3})}\right)=$$

$$=\frac{x}{18}\left(\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n+\sum_{k=0}^{\infty}\left(-\frac{x}{3}\right)^k\right)=$$

$$=\frac{x}{18}\left(\sum_{n=0}^{\infty}2\left(\frac{x}{3}\right)^{2n}\right)=\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{2n}$$

$D(f)=$ područje konvergencije $\Rightarrow \left|\frac{x}{3}\right|<1 \Rightarrow |x|<3$

Rubovi: $x=\pm 3 \Rightarrow \frac{1}{9}\sum_{n=0}^{\infty}1^{2n}$ divergira.