MATEMATIKA 2

PISMENI ISPITI 2005. – RJEŠENJA

22. siječnja
7. veljače
21. veljače
12. ožujka
9. travnja
21. lipnja
5. srpnja
13. srpnja
12. rujna
26. rujna
3. listopada
12. studenog

MAT II, rješenja, 22.01.2005.

1. Odrediti domenu od $f(x,y)=\sqrt{(1-(y-x^2))(\sin(x^2+y^2))}$

Rj. zbog $\sqrt{}$: $(1-(y-x^2))\sin(x^2+y^2)\geq 0$

a) $1-(y-x^2)\geq 0$ i $\sin(x^2+y^2)\geq 0$

$\Rightarrow y-x^2\leq 1$

$y\leq 1+x^2$

$\Rightarrow x^2+y^2\in[0,\pi]+2k\pi$

$\Rightarrow x^2+y^2$ je iz intervala

$\ldots[-4\pi,-3\pi]\cup[-2\pi,-\pi]\cup[0,\pi]\cup$

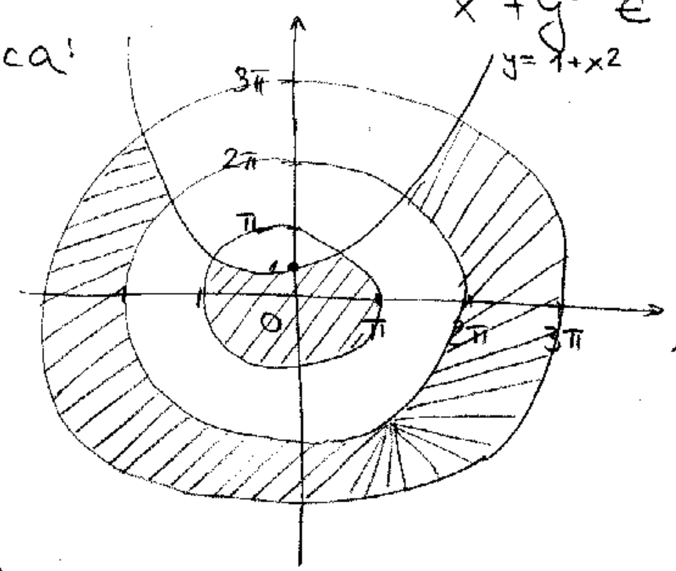
$\cup[2\pi,3\pi]\cup\ldots$

Negativni dio otpada pa

jer $x^2+y^2\geq 0$ pa ostaje

$x^2+y^2\in[0,\pi]\cup[2\pi,3\pi]\cup\ldots$

Skica:



b) $1-(y^2-x^2)\leq 0$ $\sin(x^2+y^2)\leq 0$ analogno

2. Naći tang. ravninu na plohu $2x^2+y^2+3z^2-1=0$ paralelnu sa ravninom $x+2y+3z=1$

Rj. jdn. tang. rav. u (x_0,y_0) je:

$$F_x(x_0,y_0)(X-x_0)+F_y(x_0,y_0)(Y-y_0)+F_z(x_0,y_0)(Z-z_0)=0$$

Ako je $\parallel$ sa $x+2y+3z=1$, vektori normala su kolinearni, $\vec{n_1}=\lambda\vec{n_2}$

$$\vec{n_1}=F_x(x_0,y_0)\vec{i}+F_y(x_0,y_0)\vec{j}+F_z(x_0,y_0)\vec{k}$$

22.01.05.

$\vec{n}_2 = 1 \cdot \vec{i} + 2\vec{j} + 3\vec{k}$

$\Rightarrow \quad F_x(x_0, y_0) = 4x_0 = \lambda \Rightarrow x_0 = \lambda/4$

$\quad F_y(x_0, y_0) = 2y_0 = 2\lambda \Rightarrow y_0 = \lambda$

$\quad F_z(x_0, y_0) = 6z_0 = 3\lambda \Rightarrow z_0 = \lambda/2$

pa, kako je točka na plohi, vrijedi:

$$2\left(\frac{\lambda}{4}\right)^2 + \lambda^2 + 3\left(\frac{\lambda}{2}\right)^2 - 1 = 0$$

$$\frac{\lambda^2}{8} + \lambda^2 + \frac{3\lambda^2}{4} - 1 = 0 \Rightarrow \lambda^2 + 8\lambda^2 + 6\lambda^2 = 8$$

$$\Rightarrow 15\lambda^2 = 8 \Rightarrow \lambda = \pm\frac{2\sqrt{2}}{\sqrt{15}}, \ \lambda_1 = \frac{2\sqrt{2}}{\sqrt{15}}, \ \lambda_2 = -\frac{2\sqrt{2}}{15}$$

$$T_1 = \left(\frac{\sqrt{2}}{2\sqrt{15}}, \frac{2\sqrt{2}}{15}, \frac{\sqrt{2}}{\sqrt{15}}\right), \quad T_2 = \left(-\frac{\sqrt{2}}{2\sqrt{15}}, -\frac{2\sqrt{2}}{15}, -\frac{\sqrt{2}}{\sqrt{15}}\right)$$

i sada u tim točkama imamo tang. ravnine:

$$\Pi_1 \ldots \frac{2\sqrt{2}}{\sqrt{15}}\left(x - \frac{\sqrt{2}}{2\sqrt{15}}\right) + \frac{4\sqrt{2}}{\sqrt{15}}\left(y - \frac{2\sqrt{2}}{\sqrt{15}}\right) + \frac{6\sqrt{2}}{\sqrt{15}}\left(z - \frac{\sqrt{2}}{\sqrt{15}}\right) =$$

$$\Pi_2 \ldots \frac{2\sqrt{2}}{\sqrt{15}}\left(x + \frac{\sqrt{2}}{2\sqrt{15}}\right) + \frac{4\sqrt{2}}{\sqrt{15}}\left(y + \frac{2\sqrt{2}}{\sqrt{15}}\right) + \frac{6\sqrt{2}}{\sqrt{15}}\left(z + \frac{\sqrt{2}}{\sqrt{15}}\right) =$$

3. Izračunati $\frac{\partial^2 z}{\partial y \partial x}(0,0)$ ako je $z = \frac{x+y}{x+yz+2}$.

Rj. $\quad F(x,y,z) = x + y - zx - yz^2 - 2z$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{F_x}{F_z}(x,y,z) = -\frac{1-z}{-x-2yz-2} = \frac{1-z}{x+2yz+2}$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{F_y}{F_z}(x,y,z) = \frac{1-z^2}{x+2yz+2}$$

$$\frac{\partial^2 z}{\partial y \partial x}(x,y) = \frac{-\frac{\partial z}{\partial y}(x,y)(x+2yz+2) - (1-z)\left(2z + 2y\frac{\partial z}{\partial y}(x,y)\right)}{(x+2yz+2)^2}$$

$$z(0,0) = \frac{0+0}{0+0+2} = 0 \quad \frac{\partial z}{\partial y}(0,0) = \frac{1-0}{0+2} = \frac{1}{2}$$

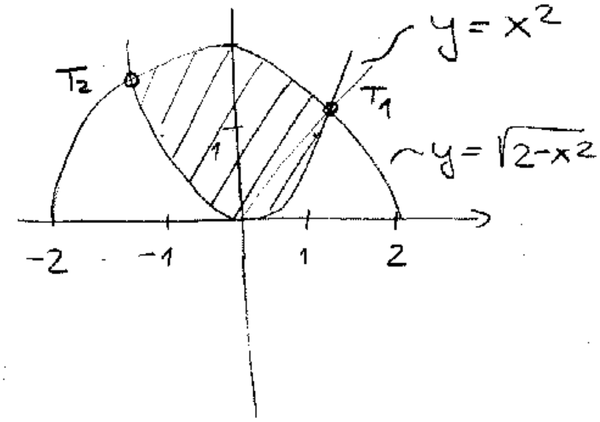
$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x}(0,0) = \frac{-\frac{1}{2}(0+0+2)}{4} = -\frac{1}{4}$$

4. Prelaskom na pol. koord. izračunati: 22.01.0

$$\int_{-1}^{1} dx \int_{x^2}^{\sqrt{2-x^2}} \frac{1}{x}\, dy$$

Rj $\quad x = r(\varphi)\cos\varphi \,, \quad y = r(\varphi)\sin\varphi$

područje integracije:



$y=\sqrt{1-x^2}\,,\ y=x^2 \Rightarrow x^4 = 1-x^2$

$\Rightarrow t^2+t-1=0 \quad t=-2,\ t=1$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$T_1(1,1) \quad T_2(-1,1)$

$\varphi_1 = \frac{\pi}{4} \qquad \varphi_2 = \frac{3\pi}{4}$

$\cancel{r}\sin\varphi = r^{\cancel{2}}\cos^2\varphi \Rightarrow r = \sin\varphi/\cos^2\varphi$ pa imamo

$$\int_{-1}^{1} dx \int_{x^2}^{\sqrt{2-x^2}} \frac{1}{x}\, dy = \int_0^{\pi/4} d\varphi \int_0^{\sin\varphi/\cos^2\varphi} \frac{1}{\cancel{r}\cos\varphi}\cancel{r}\,dr + \int_{\pi/4}^{3\pi/4} d\varphi \int_0^{\sqrt{2}} \frac{1}{\cancel{r}\cos\varphi}\cancel{r}\,dr +$$

$$+ \int_{3\pi/4}^{\pi} d\varphi \int_0^{\sin\varphi/\cos^2\varphi} \frac{1}{\cancel{r}\cos\varphi}\cancel{r}\,dr = \int_0^{\pi/4} \frac{\sin\varphi}{\cos^2\varphi}\, d\varphi + \int_{\pi/4} \sqrt{2}\, d\varphi + \int_{3\pi/4}^{\pi} \frac{\sin\varphi}{\cos^2\varphi}\, d\varphi =$$

$$= \frac{1}{\cos\varphi}\Big|_0^{\pi/4} + \sqrt{2}\cdot\frac{\pi}{2} + \frac{1}{\cos\varphi}\Big|_{3\pi/4}^{\pi} = \left(\frac{1}{\frac{\sqrt{2}}{2}} - 1\right) + \frac{\sqrt{2}}{2}\pi + \left(-1 + \frac{1}{\frac{\sqrt{2}}{2}}\right)$$

$$= \frac{2}{\sqrt{2}} - 1 + \frac{\sqrt{2}}{2}\pi - 1 - \frac{2}{\sqrt{2}} = \frac{\sqrt{2}}{2}\pi + \frac{4}{\sqrt{2}} - 2 = \frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 2$$

5. Riješiti C-problem: $y'' - y = 2x\sin x \,, \quad y(0) = 0, \ y'(0) = 1$

Rj: homogena: $y'' - y = 0$

$\lambda^2 - 1 = 0 \,, \quad \lambda = \pm 1$

22.01.05

$\Rightarrow$ rješenje hom. je $y = C_1 e^{x} + C_2 e^{-x}$

Tražimo partikularno rješ., $f(x) = 2x\sin x$

$$f(x) = e^{ax}\left[P_n(x)\cos bx + Q_m(x)\sin bx\right]$$

$$\Rightarrow \quad a = 0, \quad P_n(x) = 0, \quad Q_m(x) = 2x, \quad b = 1$$

$a + bi = i$ i to nije rješenje hom. jedn.

pa stavljamo

$$Y_p = (Ax+B)\cos x + (Cx+D)\sin x$$

$$Y_p' = A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x =$$

$$= (Cx+A+D)\cos x - (Ax+B-C)\sin x$$

$$Y_p'' = C\cos x - (Cx+A+D)\sin x - A\sin x - (Ax+B-C)\cos x$$

$$\Rightarrow (C - Ax - B + C)\cos x - (Cx + A + D + A)\sin x - (Ax+B)\cos x - (Cx+D)\sin x = 2x\sin x$$

$$\Rightarrow 2C - 2Ax - 2B = 0 \Rightarrow A = 0, \; C = B$$

$$2A + 2D = 0$$

$$-2C = 2 \Rightarrow C = -1, \; B = -1, \; D = 0$$

$$\Rightarrow Y_p = -\cos x - x\sin x$$

$$Y = y + Y_p = C_1 e^{x} + C_2 e^{-x} - \cos x - x\sin x$$

$$Y(0) = C_1 + C_2 - 1$$

$$Y' = C_1 e^{x} - C_2 e^{-x} + \cancel{\sin x} - \cancel{\sin x} - x\cos x =$$

$$= C_1 e^{x} - C_2 e^{-x} - x\cos x$$

$\Rightarrow Y'(0) = C_1 - C_2$

pa imamo

$$\left.\begin{array}{l} C_1 + C_2 - 1 = 0 \\ C_1 - C_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} 2C_1 - 1 = 1 \\ C_1 = 1 \\ C_2 = 0 \end{array}$$

$\Rightarrow Y_c = e^x - \cos x - x \sin x$

je traženo rješenje C- problema.

1. Odredite $\lambda \in \mathbb{R}$ t.d. $D(f)$ bude $[0,+\infty) \times \mathbb{R}$ ako je f zadana sa

$$f(x,y) = \sqrt{(\lambda+1)x(x^2+y^2+1)} + \arcsin \frac{\lambda}{y^2+1}$$

Rj zbog $\sqrt{}$: $(\lambda+1)x\underbrace{(x^2+y^2+1)}_{\geq 0} \geq 0 \Rightarrow (\lambda+1)x \geq 0$

Jer $x \in [0,+\infty) \Rightarrow (\lambda+1) \geq 0 \Rightarrow \lambda \geq -1$

zbog $\arcsin \frac{\lambda}{y^2+1} \Rightarrow -1 \leq \frac{\lambda}{y^2+1} \leq 1 \Rightarrow$ jer $y^2+1>0$, štoviše $y^2+1>1 \Rightarrow -1 \leq \lambda \leq 1$ (to je najgora moguća varijanta, kada $y=0$. Ako npr. $y=1 \Rightarrow -1 \leq \frac{\lambda}{2} \leq 1$ $\Rightarrow -2 \leq \lambda \leq 2$. Općenito $-1 (y^2+1) \leq \lambda \leq y^2+1$ pa je λ najviše ograničen upravo za y^2+1 minim tj. $y^2+1=1$)

$[-1,1] \cap [-1,+\infty) = [-1,1] \Rightarrow \lambda \in [-1,1]$

2. Odredite površinu omeđenu krivuljom $y = xe^{-x^2/2}$ i njenom horiz. asimptotom (skica!)

Rj $\lim\limits_{x \to \pm\infty} \frac{x}{e^{x^2/2}} = L'H = 0_{\pm}$

$y'(x) = e^{-x^2/2} + xe^{-x^2/2} \cdot (-2x/2) = e^{-\frac{x^2}{2}}(1 + x(-x))$

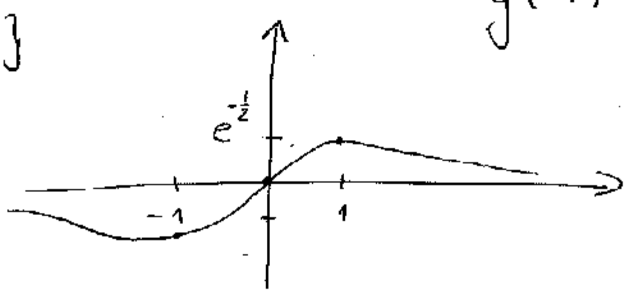
$\Rightarrow y' = 0$ za $1-x^2=0 \Rightarrow x = \pm 1$ ekstremi tj. kandid za ekstreme

$y''(x) = -xe^{-\frac{x^2}{2}}(1-x^2) + e^{-\frac{x^2}{2}}(1-2x)$

$y''(1) = e^{-\frac{1}{2}}(-1) = -e^{-\frac{1}{2}} \Rightarrow x=1$ je lok. max
$y(1) = e^{-1/2}$

$y''(-1) = e^{-\frac{1}{2}} \cdot 3 \Rightarrow x=-1$ je lok. min.
$y(-1) = -e^{-\frac{1}{2}}$

$N(f) = \{0\}$



skica grafa

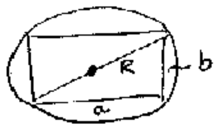
$$P = (\text{zbog neparnosti fkcije}) = 2\int_0^{+\infty} xe^{-x^2/2}\,dx =$$

$$= \lim_{b\to+\infty} 2\int_0^{b} xe^{-x^2/2}\,dx = 2\lim_{b\to+\infty}\left(-e^{-\frac{x^2}{2}}\Big|_0^b\right) =$$

$$= 2\lim_{b\to+\infty}\left(\underbrace{-e^{-\frac{b^2}{2}}}_{\to 0} + e\right) = 2e$$

3. Izračunajte približno polumjer kružnice opisane pravokutn[iku] stranica $a = 4.86$ i $b = 12.4$

Rj



očito $2R = \sqrt{a^2+b^2}$

$$R = \frac{\sqrt{a^2+b^2}}{2}$$

$$f(a,b) = \frac{\sqrt{a^2+b^2}}{2}$$

$$f(a_0+\Delta a, b_0+\Delta b) \approx \frac{\partial f}{\partial a}(a_0)\Delta a + \frac{\partial f}{\partial b}(b_0)\Delta b + f(a_0,b_0)$$

Stavimo $a_0 = 5 \Rightarrow a_0 + \Delta a = 4.86 \Rightarrow \Delta a = -0.14$ $\quad T = (a_0, b_0)$

$b_0 = 12 \Rightarrow b_0 + \Delta b = 12.4 \Rightarrow \Delta b = 0.4$

$$f(4.86, 12.4) \approx \frac{\partial f}{\partial a}(T)\cdot(-0.14) + \frac{\partial f}{\partial b}(T)\,0.4 + f(5,12)$$

$$\frac{\partial f}{\partial a}(T) = \frac{1}{\cancel{2}}\,\frac{1}{\sqrt{a^2+b^2}}\cdot\frac{1}{2}\cdot\cancel{2}a = \frac{1}{2}\,\frac{a}{\sqrt{a^2+b^2}}$$

$$\frac{\partial f}{\partial b}(T) = \frac{1}{2}\,\frac{b}{\sqrt{a^2+b^2}}$$

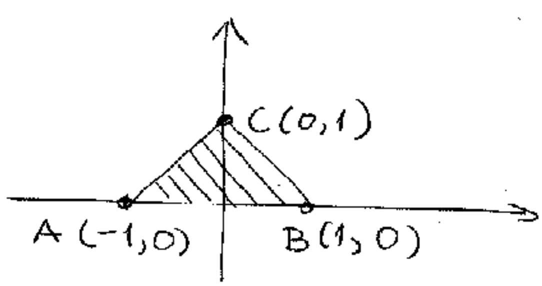
$$f(5,12) = \frac{\sqrt{25+144}}{2} = \frac{\sqrt{169}}{2} = 13/2$$

$$\frac{\partial f}{\partial a}(T) = \frac{1}{2}\,\frac{5}{\sqrt{25+144}} = \frac{1}{2}\cdot\frac{5}{13}, \quad \frac{\partial f}{\partial b}(T) = \frac{1}{2}\,\frac{12}{13}$$

$$\Rightarrow f(4.86, 12.4) \approx \frac{5}{26}\cdot\left(-\frac{14}{100}\right) + \frac{12}{26}\cdot 0.4 + 13$$

4. Izračunajte integral $\iint\limits_D xy\,dx\,dy$ gdje je područje integracije D trokut sa vrhovima $A(-1,0)$, $B(1,0)$ i $C(0,1)$.

Rj



pravac AC: $y_1 = x+1$, pravac BC: $y_2 = -x+1$

$x_1 = y_1 - 1$ $\qquad$ $x_2 = 1 - y_2$

$$\iint_D xy\,dx\,dy = \int_0^1 dy \int_{y-1}^{1-y} xy\,dx = \int_0^1 \left(\frac{1}{2}x^2 y \Big|_{y-1}^{1-y}\right) dy =$$

$$= \int_0^1 \left(\frac{1}{2}(1-y)^2 y - \frac{1}{2}(y-1)^2 y\right) dy = 0$$

5. Razvijte u Taylorov red oko 0 fkciju $f(x) = \frac{1}{x^2-3x-4}$ i nađite $f^{(100)}(0)$.

Rj: $x^2-3x-4 = (x-4)(x+1)$ pa imamo

$$\frac{1}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1} = \frac{Ax+A+Bx-4B}{x^2-3x-4}$$

$$(A+B)=0 \Rightarrow A=-B$$
$$A-4B=1 \qquad -5B=1 \Rightarrow B=-\frac{1}{5},\ A=\frac{1}{5}$$

$$f(x) = \frac{1}{x^2-3x-4} = \frac{1}{5}\left(\frac{1}{x-4} - \frac{1}{x+1}\right) = \frac{1}{5}\left(-\frac{1}{4}\left(\frac{1}{1-\frac{x}{4}}\right) - \frac{1}{1-(-x)}\right) = -\frac{1}{5}\left(\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{x}{4}\right)^k + \sum_{k=0}^{\infty}(-x)^k\right) =$$

$$= -\frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{1}{4}\cdot\frac{1}{4^k}x^k + (-1)^k x^k\right) =$$

$$= -\frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{1}{4^{k+1}} + (-1)^k\right)x^k$$ je razvoj u Taylorov red oko nule

Znamo: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$

$$\Rightarrow \frac{f^{(100)}}{100!} = -\frac{1}{5}\left(\frac{1}{4^{101}} + 1\right)$$

$$\Rightarrow f^{(100)} = -\frac{1}{5}\left(\frac{1}{4^{101}} + 1\right)\cdot 100!$$

Konvergencija: $\left|\frac{x}{4}\right| < 1$ i $|x| < 1 \Rightarrow |x| < 1$

MAT I RJEŠENJA, 21.02.2005.

1. Odredite domenu fkcije zadane sa $f = g \circ h$ ako je $h(x,y) = \sqrt{x^2+y^2-1}$ a $g(t) = \arcsin t$.

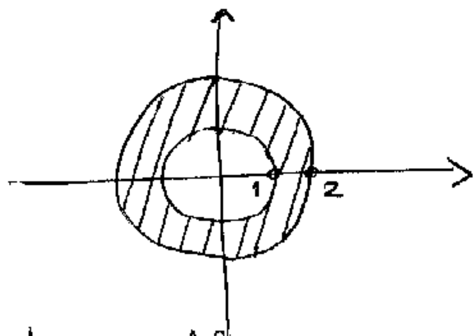
Rj

$$f(x,y) = g \circ h(x,y) = g\left(\sqrt{x^2+y^2-1}\right) =$$
$$= \arcsin\left(\sqrt{x^2+y^2-1}\right)$$

zbog arcsin: $\underset{\substack{\uparrow \\ \text{automatski vrijedi}}}{-1 \leq} \sqrt{x^2+y^2-1} \leq 1$

$$\Rightarrow \sqrt{x^2+y^2-1} \leq 1 \Rightarrow x^2+y^2-1 \leq 1 \Rightarrow x^2+y^2 \leq 2$$

zbog $\sqrt{}$: $x^2+y^2 \geq 1 \Rightarrow 1 \leq x^2+y^2 \leq 2$



2. Naći lok. ekstreme fkcije z zadane sa
$z^3 - 3z(x^2+y^2) - 27 = 0$

Rj $F(x,y,z) = z^3 - 3z(x^2+y^2) - 27$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{F_x}{F_z}(x,y,z) = -\frac{6zx}{3z^2-3(x^2+y^2)}$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{F_y}{F_z}(x,y,z) = -\frac{6zy}{3z^2-3(x^2+y^2)}$$

$$\frac{\partial z}{\partial x}(x,y) = \frac{\partial z}{\partial y}(x,y) = 0 \Rightarrow zx = zy = 0 \Rightarrow x = y = 0$$

$z(0,0)$: $z(0,0)^3 - 0 - 27 = 0 \Rightarrow z^3(0,0) = 27$
$\Rightarrow z(0,0) = 3$

$zx = zy = 0$ može još biti ako $z(x,y) = 0$
No, onda imamo $0^3 - 3 \cdot 0 - 27 = 0$ što ne može.

Jedini kandidat: $(x,y) = (0,0)$

$$\frac{\partial^2 z}{\partial x^2}(x,y) = -6\,\frac{\left(z+\frac{\partial z}{\partial x}x\right)\left(3z^2-3(x^2+y^2)\right) - zx(\quad)}{\left(3z^2-3(x^2+y^2)\right)^2}$$

nije bitno jer gledamo $\frac{\partial^2 z}{\partial x^2}(0,0)$ pa $x=0$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{0}{27} = 0 = \frac{\partial z}{\partial y}(x,y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}(x,y) = -6\cdot\frac{(3+0)\cdot\cancel{27}}{27^{\cancel{2}}} = -\frac{18}{27}$$

$$\frac{\partial^2 z}{\partial y^2}(x,y) = -6\,\frac{\left(z+\frac{\partial z}{\partial y}y\right)\left(3z^2-3(x^2+y^2)\right) - zy(\quad)}{\left(3z^2-3(x^2+y^2)\right)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2}(0,0) = -\frac{6\cdot 3\cdot 27}{27^2} = -\frac{18}{27}$$

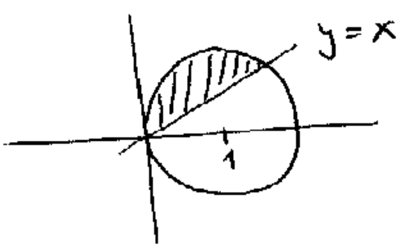
$$\frac{\partial^2 z}{\partial y\,\partial x}(x,y) = -\frac{6\left(x\frac{\partial z}{\partial y}\right)\left(3z^2-3(x^2+y^2)\right) - 6zx(\quad)}{\left(3z^2-3(x^2+y^2)\right)^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial y\,\partial x}(0,0) = 0$$

$\Rightarrow \det = -\frac{18}{27}\left(-\frac{18}{27}\right) - 0 > 0 \Rightarrow$ to je lok. ekstr. i to lok. max

3. Primjenom pol. koord. izračunati $\iint_D (x^2+y^2)\,dx\,dy$ ako je D gornji podr. omeđen sa $(x-1)^2+y^2=1$ i $y=x$.

Rj.



$$x^2-2x+y^2=0$$

$$r^2\cos^2\varphi - 2r\cos\varphi + r^2\sin^2\varphi = 0$$

$$\Rightarrow r^2 = 2r\cos\varphi$$

$$r = 2\cos\varphi$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\cdot r\,dr = \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \left(r^4\Big|_0^{2\cos\varphi}\right) =$$

$$= \frac{1}{4}\cdot 16 \int_{\frac{\pi}{4}}^{\pi/2} \cos^4\varphi\,d\varphi = 4\int_{\frac{\pi}{4}}^{\pi/2}\left(\frac{1}{2}(1+\cos 2\varphi)\right)^2 d\varphi =$$

$$= \int_{\pi/4}^{\pi/2} \left(1 + 2\cos 2\varphi + \frac{1}{2}(1+\cos 4\varphi)\right) d\varphi =$$

$$= \left(\varphi + \sin 2\varphi + \frac{1}{2}\varphi + \frac{1}{8}\sin 4\varphi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right) =$$

$$= \frac{\pi}{2} + 0 + \cancel{\frac{1}{2}\cdot\frac{\pi}{2}} + 0 - \cancel{\frac{\pi}{4}} - 1 - \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{8}\cdot 0 =$$

$$= \frac{4\pi}{8} - \frac{\pi}{8} - 1 = \frac{3\pi}{8} - 1$$

4. Među svim kvadrima volumena a nađite onaj najmanjeg oplošja.

Rj. $V = xyz = a \Rightarrow z = \frac{a}{xy}$

$$O = 2xy + 2xz + 2yz = 2xy + 2x\frac{a}{xy} + 2y\frac{a}{xy} =$$

$$= 2\left(xy + \frac{a}{y} + \frac{a}{x}\right)$$

$$\frac{\partial V}{\partial x}(x,y) = 2\left(y - \frac{a}{x^2}\right), \quad \frac{\partial V}{\partial y}(x,y) = 2\left(x - \frac{a}{y^2}\right)$$

$$\Rightarrow y - \frac{a}{x^2} = 0 = x - \frac{a}{y^2}$$

$$yx^2 = a = y^2 x \Rightarrow x = \frac{a}{y^2}$$

$$y\cdot\left(\frac{a}{y^2}\right)^2 = a \Rightarrow \frac{a^2}{y^3} = a \Rightarrow \frac{a}{y^3} = 1 \Rightarrow y = \sqrt[3]{a}$$

$$x = \sqrt[3]{a}$$

i sada idu druge derivacije za provjeru.

5. Razviti u T-red oko $x = 2$ funkciju $f(x) = \frac{1}{5-x}$ i naći konv, $f^{(50)}(2)$.

Rj. $$f(x) = \frac{1}{5-x} = \frac{1}{3+2-x} = \frac{1}{3-(x-2)} = \frac{1}{3}\frac{1}{1-\left(\frac{x-2}{3}\right)} = \text{geom red}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x-2}{3}\right)^n = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x-2)^n, \text{ konv. } \left|\frac{x-2}{3}\right| < 1$$

$$\Rightarrow |x-2| < 3$$

$$\Rightarrow x \in (1, 5)$$

$$f^{(50)}(20) = 1/3^{51}.$$

1) $D(f)$, f zadana sa $f = g \circ h$, $h(x,y) = \ln(y-x^2)$
$g(t) = \sqrt{t}$. Skica!

Rj $\quad f = g \circ h \Rightarrow f(x,y) = \sqrt{\ln(y-x^2)}$

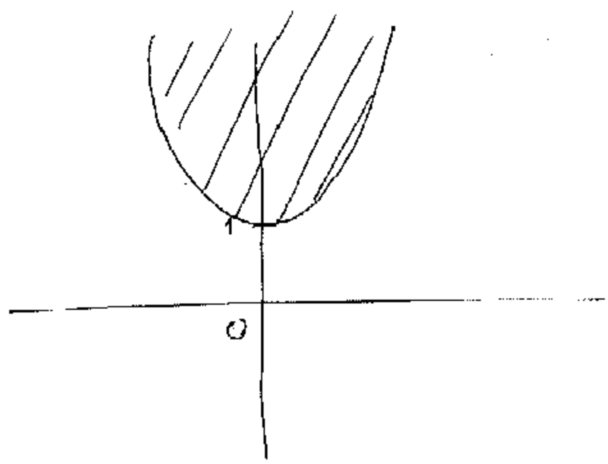
zbog $\sqrt{}$: $\ln(y-x^2) \geq 0 \quad / e$

$\Rightarrow y - x^2 \geq e^0 = 1$

$y \geq 1 + x^2$

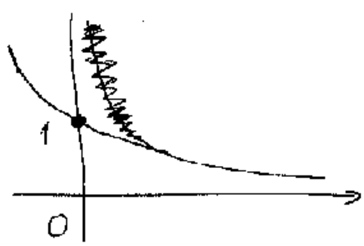
zbog ln : $y - x^2 > 0$

Sve skupa : $y - x^2 \geq 1$ tj. $y \geq 1 + x^2$



2) Volumen tijela nastalog rotacijom $y = e^{-x}$, $x \geq 0$ oko osi OX.

Rj



očito je u pitanju nepravi integral : $\int_0^{+\infty} \pi y^2 dx =$

$$= \int_0^{+\infty} \pi (e^{-x})^2 dx = \pi \int_0^{+\infty} e^{-2x} dx = -\frac{\pi}{2} \lim_{b \to +\infty} \left(e^{-2x} \Big|_0^b \right)$$

$$= -\frac{\pi}{2} \lim_{b \to +\infty} \Big(\underbrace{\frac{1}{e^{2b}}}_{=0} - \underbrace{\frac{1}{e^0}}_{=1} \Big) = \frac{\pi}{2}$$

3) Na sferi $x^2+y^2+z^2=1$ odredite tang. ravninu koja sadrži pravac $\dfrac{x-\frac{\sqrt{2}}{2}}{2}=\dfrac{y-\frac{\sqrt{2}}{2}}{-2}=\dfrac{z}{0}$ $\underline{\underline{2.}}$

$\underline{Rj}$ ~~pro~~ provjeravamo da li se pravac i sfera sijeku:

$$x=2t+\frac{\sqrt{2}}{2},\quad y=-2t+\frac{\sqrt{2}}{2},\quad z=0$$

$$x^2+y^2+z^2=1 \Rightarrow \left(2t+\frac{\sqrt{2}}{2}\right)^2+\left(-2t+\frac{\sqrt{2}}{2}\right)^2+0=1$$

$$4t^2+\cancel{2\sqrt{2}t}+\cancel{\frac{1}{2}}+4t^2-\cancel{2\sqrt{2}t}+\cancel{\frac{1}{2}}=1$$

$$8t^2=0 \Rightarrow t=0$$

pa ima samo jedno presjecište, $x=\frac{\sqrt{2}}{2}, y=\frac{\sqrt{2}}{2}, z=0$ i, jer je riječ o sferi, možemo zaključiti da je taj pravac tangenta, pa automatski leži u tang. ravnini na točku $M\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)$.

Jd. tang. ravn. u $T(x_0,y_0,z_0)$:

$$\frac{\partial F}{\partial x}(T)(X-x_0)+\frac{\partial F}{\partial y}(T)(Y-y_0)+\frac{\partial F}{\partial z}(T)(Z-z_0)=0$$

U našem slučaju: $$\frac{\partial F}{\partial x}(M)=2x_0=2\frac{\sqrt{2}}{2}=\sqrt{2}$$

$$\frac{\partial F}{\partial y}(M)=2y_0=\sqrt{2}$$

$$\frac{\partial F}{\partial z}(M)=2z_0=0$$

$\Rightarrow \sqrt{2}\left(X-\frac{\sqrt{2}}{2}\right)+\sqrt{2}\left(Y-\frac{\sqrt{2}}{2}\right)=0$ je tražena ravnina.

4) Među svim trokutima opsega 4 naći onaj najveće površine.

$\underline{Rj}$ $O=4 \Rightarrow a+b+c=4,\quad s=\dfrac{a+b+c}{2}=\dfrac{4}{2}=2$

$$P=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{2(2-a)(2-b)(2-c)}$$

$$c=4-a-b \Rightarrow 2-c=2-4+a+b=a+b-2$$

$P(a,b) = \sqrt{2(2-a)(2-b)(a+b-2)} =$

$= \sqrt{2}\sqrt{(4-2b-2a+ab)(a+b-2)} =$

$= \sqrt{2}\sqrt{4a+4b-8-2ab-2b^2+4b-2a^2-2ab+4a+a^2b+ab^2-2ab} =$

$= \sqrt{2}\sqrt{8a+8b-2a^2-2b^2-6ab+a^2b+ab^2-8}$

$$\frac{\partial P}{\partial a}(a,b) = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{8a+8b-2a^2\ldots}}\cdot(8-4a-6b+2ab+b^2)$$

$$\frac{\partial P}{\partial b}(a,b) = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{8a+8b-2a^2\ldots}}(8-4b-6a+a^2+2ab)$$

Kandidati su: $8-4a-6b+2ab+b^2 = 8-4b-6a+a^2+2ab = 0$

$-4a-6b+b^2 = -6a-4b+a^2$

$6a-6b+4b-4a = a^2-b^2$

$2a-2b = a^2-b^2$

$2(a-b) = (a-b)(a+b)$

pro rješenje: $a=b$

$a \neq b \Rightarrow 2 = a+b$

$a=b$ $\quad 8-4a-6a+2a^2+a^2=0$

$3a^2-10a+8=0$

$$a = \frac{10 \pm \sqrt{100-4\cdot 8\cdot 3}}{6} = \frac{10\pm\sqrt{4}}{6}$$

$$a_1 = \frac{10+2}{6} = 2, \quad a_2 = \frac{10-2}{6} = \frac{8}{6} = \frac{4}{3}$$

$a=b=2$ i opseg $= a+b+c=4 \Rightarrow c=0$ ↯

Ostaje $a=b=\frac{4}{3} \Rightarrow c = \frac{12}{3}-\frac{8}{3}=\frac{4}{3}$

Provjeravamo opciju $2 = a + b \Rightarrow b = 2 - a$ 4.

$$8 - 4a - 6(2-a) + 2a(2-a) + (2-a)^2 = 0$$

$$\cancel{8} - \cancel{4a} - \cancel{12} + 6a + \cancel{4a} - 2a^2 + \cancel{4} - 4a + a^2 = 0$$

$$-a^2 + 2a = 0$$

$$a(2-a) = 0$$

ili $a = 0$ ↯ ili $a = 2 \Rightarrow b = 0$ ↯

Stoga je jedini kandidat $a = b = c = \frac{4}{3}$

Druga derivacija za provjeru!

) Razvoj u Taylorov red oko $x = 3$ od $f(x) = \frac{1}{6-x}$, konv. i $f^{(55)}(3)$.

Rj. $f(x) = \frac{1}{6-x} = \frac{1}{3+3-x} = \frac{1}{3-(x-3)} = \frac{1}{3} \frac{1}{1 - \frac{x-3}{3}} =$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x-3}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} (x-3)^n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} (x-3)^n$$

konvergencija: $\left|\frac{x-3}{3}\right| < 1 \Rightarrow |x-3| < 3 \Rightarrow$ ~~[illegible]~~ $x \in (0, 6)$

$$\frac{f^{(55)}(3)}{3!} = \frac{1}{3^{55+1}} \Rightarrow f^{(55)}(3) = \frac{3!}{3^{56}}.$$

MAT II, RJEŠENJA, 9.04.2005.

1. $D(f)$, $f = g \circ h$, $h(x,y) = \sin(x+y)$, $g(t) = \sqrt{t}\ln t$

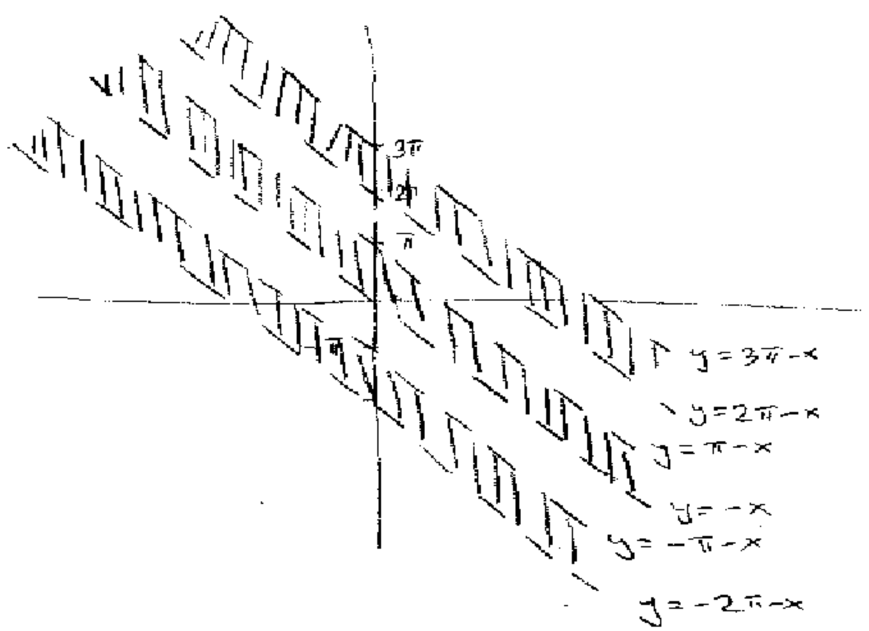
Rj $f(x,y) = g\circ h(x,y) = g(\sin(x+y)) = \sqrt{\sin(x+y)}\ln(\sin(x+y))$

zbog $\sqrt{\ }$: $\sin(x+y) \geq 0$
ln : $\sin(x+y) > 0$

$\sin(x+y) > 0 \Rightarrow x+y \in \ldots (-2\pi,-\pi)\cup(0,\pi)\cup(2\pi,3\pi)\cup\ldots$
$= (0,\pi) + 2k\pi,\ k\in\mathbb{Z}$

$x+y \in (0,\pi) \Rightarrow 0 < x+y < \pi$
$\Rightarrow -x < y$ i $y < \pi - x$

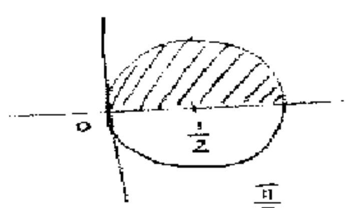
analogno ostali intervali. Skica:



2. $\iint_D f(x,y)\,dx\,dy$, D = područje u I kvadrantu omeđeno s kružnicom $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$ i x-osi, $f(x,y)=\sqrt{1-x^2-y^2}$

Rj polarne koord: $f(r,\varphi) = \sqrt{1-r^2}$

D:



$x^2 - x + \cancel{\frac{1}{4}} + y^2 = \cancel{\frac{1}{4}}$
$r^2 = r\cos\varphi \Rightarrow r(\varphi) = \cos\varphi$ jdn. kruž.

$$\iint_D f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{1-r^2}\, r\,dr = \int_0^{\frac{\pi}{2}}\left(\frac{2}{3}(1-r^2)^{\frac{3}{2}}\left(-\frac{1}{2}\right)\Big|_0^{\cos\varphi}\right)d\varphi =$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}\left((1-\cos^2\varphi)^{3/2} - 1\right)d\varphi = -\frac{1}{3}\int_0^{\frac{\pi}{2}}\sin^3\varphi\,d\varphi + \frac{1}{3}\int_0^{\frac{\pi}{2}}d\varphi =$$

$$= -\frac{1}{3}\left(\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi - \int_0^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi\, d\varphi\right) + \frac{\pi}{6}$$

$$\frac{\pi}{6} - \frac{1}{3}\left(-\cos\varphi \Big|_0^{\frac{\pi}{2}} + \frac{1}{3}\cos^3\varphi \Big|_0^{\frac{\pi}{2}}\right) = -\frac{1}{3}\left(1 + \frac{1}{3}(0-1)\right) + \frac{\pi}{6} = -\frac{2}{9} + \frac{\pi}{6}$$

3. $x^2+y^2+z^2=1$, odrediti tang. ravninu koja sadrži pravac

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{0}$$

Rj pravac : $x = t+1$, $y = t+1$, $z = 1$,, tražimo presjek sa sferom

$$(t+1)^2 + (t+1)^2 + 1 = 1 \Rightarrow 2(t+1)^2 = 0 \Rightarrow t = -1$$

pa je presj. $S(0,0,1) \Rightarrow$ samo jedna točka pa je naš pravac baš tang. u toj točki $\Rightarrow$ tražimo tang. ravninu na sferu u S:

$$\frac{\partial F}{\partial x}(S)(X-x_S) + \frac{\partial F}{\partial y}(S)(Y-Y_S) + \frac{\partial F}{\partial z}(S)(Z-z_S) = 0$$

gdje je $F = x^2+y^2+z^2-1$

$$\Rightarrow 2x(S)(X-x_S) + 2y(S)(Y-Y_S) + 2z(S)(Z-z_S) = 0$$

$2(z-1) = 0 \Rightarrow z-1=0$ tj. $z=1$ je tražena ravnina.

4. $x = z^2+y^2$, tražimo točku najbližu točki $A(6,4,2)$.

Rj

$$d(T,A) = \sqrt{(x_T-6)^2 + (y_T-4)^2 + (z_T-2)^2}$$

$x_T = z_T^2 + y_T^2$ jer je točka na plohi

$$d(T,A) = \sqrt{(y_T^2+z_T^2-6)^2 + (y_T-4)^2 + (z_T-2)^2} = f(y,z)$$

$\frac{\partial f}{\partial y}(y,z) = \frac{\partial f}{\partial z}(y,z) = 0$ da bi točka bila kand. za ekstr.

$$\Rightarrow \frac{\partial f}{\partial y}(y,z) = \frac{1}{2}\frac{1}{\sqrt{}} \cdot 2(y_T^2+z_T^2-6)\cdot 2y_T + 2(y_T-4) = 0$$

9.04.05

$$\frac{\partial f}{\partial z}(y,z)=\frac{1}{2}\frac{1}{\sqrt{}}\cdot 2(y_T^2+z_T^2-6)2z_T+2(z_T-2)=0$$

$$\Rightarrow 2(y_T^2+z_T^2-6)y_T+y_T\cdot 4=0 \quad i$$
$$2(y_T^2+z_T^2-6)z_T+z_T-2=0$$

$$\Rightarrow 2(y_T^2+z_T^2-6)(y_T-z_T)+y_T-z_T-4+2=0$$
$$2(y_T^2+z_T^2-6)(y_T-z_T)+y_T-z_T=2$$
$$(y_T-z_T)(2(y_T^2+z_T^2-6)+1)=2$$

analogno $(y_T+z_T)(2(y_T^2+z_T^2-6)+1)=6$

$$\Rightarrow \frac{y_T+z_T}{y_T-z_T}=3 \Rightarrow y_T+z_T=3y_T-3z_T$$
$$-2y_T=-4z_T$$
$$y_T=2z_T$$

$$\Rightarrow 2(4z_T^2+z_T^2-6)z_T+z_T-2=0$$
$$2(5z_T^2-6)z_T+z_T-2=0$$
$$10z_T^3-12z_T+z_T-2=0$$
$$10z_T^3-11z_T-2=0$$

Ova jednadžba ima tri rješenja koja se ne mogu dobiti elem. putem već se mora ići numerički.

<u>5.</u> Taylor oko nule od $f(x)=\frac{1}{x^2-7x+6}$, konv, $f^{(100)}(0)$

Rj. $x^2-7x+6=0 \Rightarrow x^2-6x-x+6=0 \quad x(x-6)-(x-6)=0$

$$(x-6)(x-1)=0$$

$$\frac{1}{x^2-7x+6}=\frac{A}{x-6}+\frac{B}{x-1} \Rightarrow A+B=0,\ -A-6B=1 \Rightarrow B=-\frac{1}{5},\ A=\frac{1}{5}$$

$$f(x)\ \frac{1}{5}\frac{1}{(-6)(1-\frac{x}{6})}-\frac{1}{5}\frac{1}{(-1)(1-x)}=-\frac{1}{30}\frac{1}{1-\frac{x}{6}}+\frac{1}{5}\cdot\frac{1}{1-x}$$

$$\Rightarrow f(x)=-\frac{1}{30}\sum_{n=0}^{\infty}\frac{x^n}{6^n}+\frac{1}{5}\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}\left(-\frac{1}{30\cdot 6^n}+\frac{1}{5}\right)x^n$$

konv: $\left|\frac{x}{6}\right|<1$ i $|x|<1 \Rightarrow |x|<1$, $f^{(100)}(0)=100!\left(-\frac{1}{30\cdot 6^{100}}+\frac{1}{5}\right.$

Mat 2 - RJEŠENJA 21.06.2005.

1. Domena od $f(x,y) = \arcsin(x^2+y^2) + \ln\left(\frac{x+y+1}{x+y}\right)$

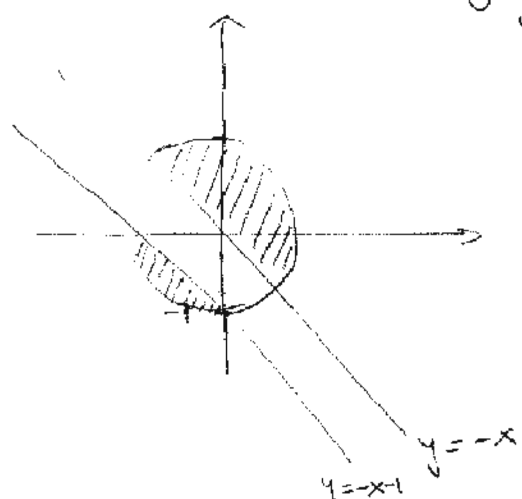
Rj arcsin: $-1 \le x^2+y^2 \le 1 \Rightarrow x^2+y^2 \le 1$

ln: $0 < \frac{x+y+1}{x+y} \Rightarrow 0 < 1 + \frac{1}{x+y} \Rightarrow \frac{1}{x+y} > -1$

$1 + \frac{1}{x+y} \neq 1 \Rightarrow \frac{1}{x+y} \neq 0$ što je uvijek

a) $x+y>0$
$1 > -x-y$
$y > -x-1$
$y > -x$

b) $x+y<0$
$1 < -x-y$
$y < -x-1$
$y < -x$



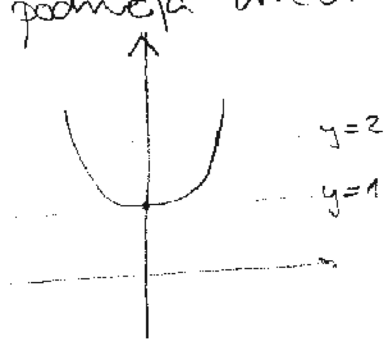
2. $\int_{\ln\frac{1}{2}}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = ?$

$$\int_{\ln\frac{1}{2}}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \lim_{\varepsilon\to 0} \int_{\ln\frac{1}{2}}^{\varepsilon} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \left| \begin{matrix} e^x = t \\ e^x\,dx = dt \end{matrix} \right| =$$

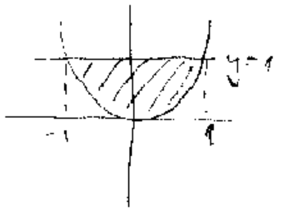
$$= \lim_{\varepsilon\to 0} \int_{\frac{1}{2}}^{e^\varepsilon} \frac{dt}{\sqrt{1-t^2}} = \lim_{\varepsilon\to 0} \left(\arcsin t \Big|_{1/2}^{e^\varepsilon} \right) = \lim_{\varepsilon\to 0} \left(\arcsin e^\varepsilon - \arcsin\frac{1}{2} \right) =$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

3. Volumen nastao rotacijom područja omeđenog sa $y = x^2+1$ i $y=2$ oko $y=1$



- To je isto kao rotacija oko $y=0$ (supst. $t=y-1$) područja omeđ. sa $y=x^2$ i $y=$



$$V=\pi\int_{-1}^{1}\left(f^2(x)-g^2(x)\right)dx=\pi\int_{-1}^{1}(1-x^4)\,dx=\pi\left(x-\frac{1}{5}x^5\Big|_{-1}^{1}\right)=$$

$$=\pi\left(1-\frac{1}{5}-\left(-1+\frac{1}{5}\right)\right)=\pi\left(2-\frac{2}{5}\right)$$

4. $\frac{\partial}{\partial x}z(x,y)$, $\frac{\partial}{\partial y}z(x,y)$ u točci $T(-1,0,z\geq 0)$ ako je z zadano sa $x^2+y^2-z^2-xy=0$

Rj

$$\frac{\partial}{\partial x}z(x,y)=-\frac{F_x}{F_z}=-\frac{2x-y}{-2z}=\frac{2x-y}{2z}$$

$1-z^2=0$

$z=1$ zbog $z\geq 0$

$$\frac{\partial}{\partial y}z(x,y)=-\frac{F_y}{F_z}=-\frac{2y-x}{-2z}=\frac{2y-x}{2z}$$

$$\Rightarrow \frac{\partial}{\partial x}z(-1,0)=\frac{2(-1)-0}{2\cdot 1}=-1,\quad \frac{\partial z}{\partial y}(-1,0)=\frac{0-(-1)}{2}=\frac{1}{2}$$

5. Taylor oko nule od $f(x)=\frac{1}{x^2-x-6}$, $f^{(51)}(0)=?$

Rj

$$x^2-x-6=x^2-3x+2x-6=x(x-3)+2(x-3)=(x-3)(x+2)$$

$$\frac{1}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}\Rightarrow \begin{array}{l}A+B=0\\ 2A-3B=1\end{array}\quad A=-B\quad -5B=1\quad B=-\frac{1}{5}\quad A=\frac{1}{5}$$

$$f(x)=\frac{1}{5}\cdot\frac{1}{x-3}-\frac{1}{5}\frac{1}{x+2}=-\frac{1}{15}\frac{1}{1-\frac{x}{3}}-\frac{1}{10}\frac{1}{1-\left(-\frac{x}{2}\right)}=$$

$$=-\frac{1}{15}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left(-\frac{1}{15}\cdot\frac{1}{3^n}-\frac{1}{10}\left(-\frac{1}{2}\right)^n\right)x^n$$

konv: $\left|\frac{x}{3}\right|<1$, $\left|\frac{x}{2}\right|<1\Rightarrow -2<x<2$

$$\frac{f^{(51)}}{51!}=-\frac{1}{15}\cdot\frac{1}{3^{51}}+\frac{1}{10}\cdot\frac{1}{2^{51}}$$

MAT II, 5/07/2005 Rješenja

1. $f(x,y)=\sqrt{6x-x^2-9-y^2+2y}+\sqrt{\sin(x+y)}$, domena

Rj $6x-x^2-9-y^2+2y=-(x^2-6x+9+y^2-2y)=$

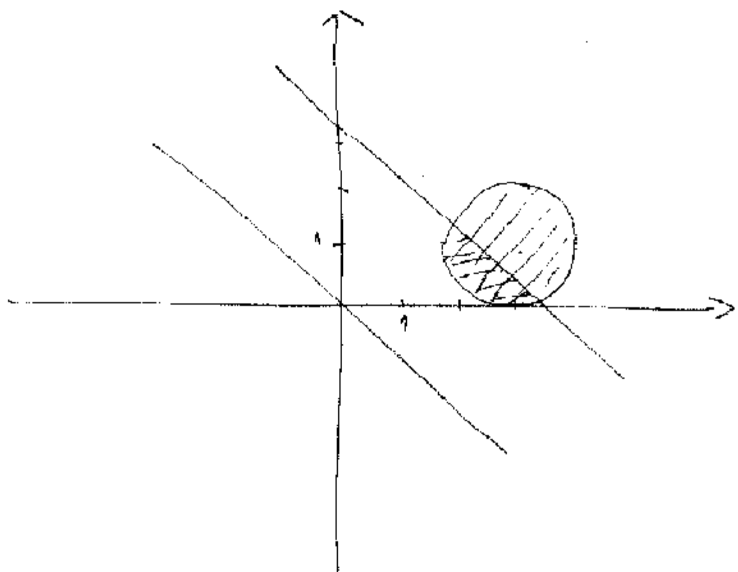
$=-\left((x-3)^2+(y-1)^2-1\right)$

zbog $\sqrt{\ }$: $(x-3)^2+(y-1)^2-1\le 0$

zbog $\sqrt{\ }$: $\sin(x+y)\ge 0 \Rightarrow x+y\in[0,\pi]+2k\pi$, $k\in\mathbb{Z}$

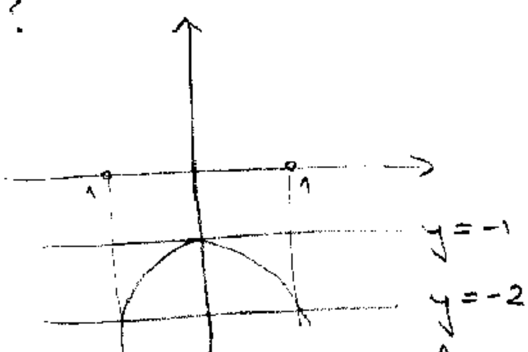
$x+y\in[0,\pi]$

$-x\le y\le -x+\pi$ itd



2. rotacija područja omeđenog krivuljama $y=-x^2-1$, $y=-2$ oko $y=-1$, volumen?

Rj



oko $y=0$

Isto kao $y=x^2$ omeđeno sa $y=1$

$$V=\int_{-1}^{1}\pi\,dx-\pi\int_{-1}^{1}(x^2)^2\,dx=\left(2-\frac{2}{5}\right)\pi=\frac{8}{5}\pi$$

3. $x^2+y^2+z^2-1=0$ $T(1,3,1)$ i $M(2,1,1)$ točke na tang ravnini.

Rj Jdu tang. ravnine u $T_0(x_0,y_0,z_0)$

$2x_0(X-x_0)+2y_0(Y-Y_0)+2z_0(Z-z_0)=0$

jer su točke u uvj., imamo:

$$2x_0(1-x_0)+2y_0(3-y_0)+2z_0(1-z_0)=0$$

$$\Rightarrow \quad x_0-x_0^2+3y_0-y_0^2+z_0-z_0^2=0$$

i $\quad 2x_0(2-x_0)+2y_0(1-y_0)+2z_0(1-z_0)=0$

$$\Rightarrow 2x_0-x_0^2+y_0-y_0^2+z_0-z_0^2=0$$

jer $\quad x_0^2+y_0^2+z_0^2=1 \quad \Longrightarrow \quad x_0+3y_0+z_0=1$

$$2x_0+y_0+z_0=1$$

$$-x_0+2y_0=0$$

$$y_0=\frac{x_0}{2}$$

$$\Rightarrow \quad 2x_0+\frac{x_0}{2}+z_0=1$$

$$z_0=1-\frac{5}{2}x_0$$

$$\Rightarrow x_0^2+\frac{x_0^2}{4}+\cancel{1}-5x_0+\frac{25}{4}x_0^2=\cancel{1}$$

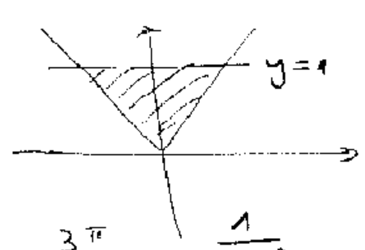
$x_0=0 \quad$ i $\quad x_0\left(1+\frac{1}{4}+\frac{25}{4}\right)=5$

$$x_0\cdot\frac{30}{4}=5 \quad \Rightarrow \quad x_0=\frac{4}{\cancel{30}_6}\cdot\cancel{5}^1=\frac{2}{3}$$

Rj su $\quad (0,0,1) \quad$ i $\quad \left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)$

4. Polarne koordinate, izračunati $\int_{-1}^{0}\int_{-x}^{1}\sqrt{x^2+y^2}\,dx\,dy+\int_{0}^{1}\int_{x}^{1}\sqrt{x^2+y^2}\,dx\,dy$

Rj



$$r\sin\varphi=1$$

$$I=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}d\varphi\int_0^{\frac{1}{\sin\varphi}}\sqrt{r^2}\,r\,dr=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}d\varphi\int_0^{1/\sin\varphi}r^2\,dr=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{1}{3}\frac{1}{\sin^3\varphi}\right)d\varphi=$$

$$\int\frac{d\varphi}{\sin^3\varphi}=\text{Demidović, str } 129=\frac{1}{4}\left(-\frac{1}{2\,\mathrm{tg}^2\frac{\varphi}{2}}+2\ln\left|\mathrm{tg}\frac{\varphi}{2}\right|+\frac{\mathrm{tg}^2\frac{\varphi}{2}}{2}\right)$$

itd, samo se uvrste brojevi.

5. Taylor oko $x_0 = 2$ za $f(x) = \frac{1}{x^2-4x}$, konvergencija. (5/07/05)

Rj. $x^2 - 4x = (x-2)^2 - 4$

$$f(x) = \frac{1}{x^2-4x} = \frac{1}{(x-2)^2-4} = -\frac{1}{4}\,\frac{1}{1-\frac{(x-2)^2}{4}} = -\frac{1}{4}\,\frac{1}{1-\left(\frac{x-2}{2}\right)^2} =$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x-2}{2}\right)^{2n} = \sum_{n=0}^{\infty} -\frac{1}{4^{2n+1}}(x-2)^{2n}$$

konv. : $\left|\frac{x-2}{2}\right|^2 < 1 \Rightarrow \left|\frac{x-2}{2}\right| < 1 \Rightarrow |x-2| < 2$

$x \in (0,4)$

MAT II, RJEŠENJA 13/07/2005

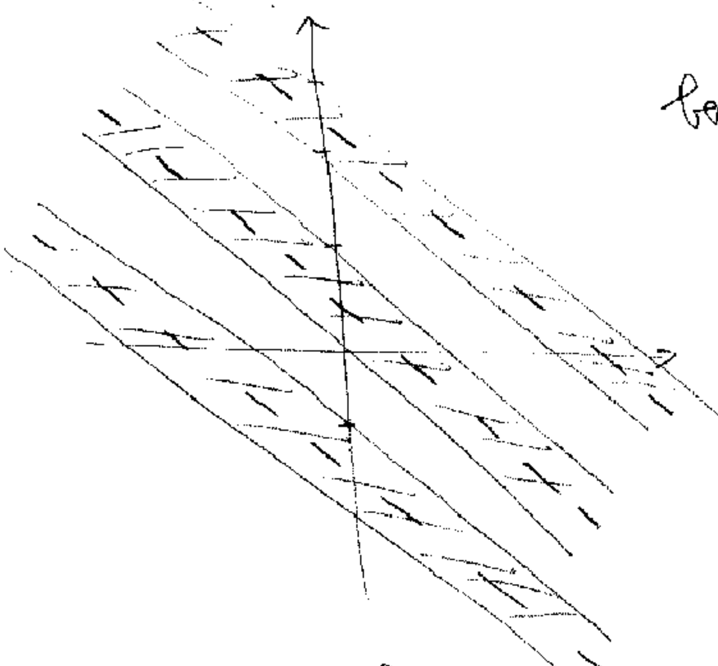
1. $D(f)$, $f = g \circ h$, $h(x,y) = \sin(x+y)$, $g(t) = \sqrt{t} + \ln(1-t)$

Rj $f(x,y) = \sqrt{\sin(x+y)} + \ln(1 - \sin(x+y))$

zbog $\sqrt{\ }$: $\sin(x+y) \geq 0 \Rightarrow x+y \in [0,\pi] + 2k\pi$

zbog $\ln$: $1 - \sin(x+y) > 0 \Rightarrow \sin(x+y) \neq 0$

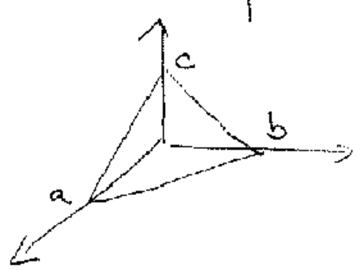
$x+y \neq \frac{\pi}{2} + 2k\pi$



bez - - - pravaca

2. $T(1,2,3)$; ravnina kroz T koja sa koord. osima zatvara piramidu najmanjeg volumena

Rj



$$V = \frac{abc}{3}$$

jdn. ravnine $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

T u ravnini $\Rightarrow \frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ $\quad bc + 2ac + 3ab = abc$

$$c = -\frac{3ab}{b+2a-ab} = \frac{3ab}{ab-b-2a}$$

$$V = \frac{a^2b^2}{ab-b-2a}$$

$$\frac{\partial V}{\partial a}(a,b) = \frac{2ab^2(ab-b-2a) - a^2b^2(b-2)}{(ab-b-2a)^2} = 0$$

$$\frac{\partial V}{\partial b}(a,b) = \frac{2a^2b(ab-b-2a) - a^2b^2(a-1)}{(ab-b-2a)^2} = 0$$

$ab \neq 0 \Rightarrow \quad 2(ab-b-2a)-a(b-2)=0$

$2(ab-b-2a)-b(a-1)=0$

$\Rightarrow a(b-2)=b(a-1)$

$\cancel{ab}-2a=\cancel{ab}-b$

$2a=b \Rightarrow a=\frac{b}{2}$

$2\left(\frac{b^2}{2}-b-b\right)-\frac{b}{2}(b-2)=0 \quad /:b$

$2\left(\frac{b}{2}-2\right)-\frac{1}{2}(b-2)=0 \Rightarrow b-4-\frac{b}{2}+1=0$

$\frac{b}{2}=3 \quad b=6$

$a=3$

$$c=\frac{3\cdot 3\cdot 6}{3\cdot 6-6-6}=\frac{9\cdot 6}{6}=9$$

3. Izračunati približno $\sqrt[3]{5.8+\sqrt[4]{16.01}}$

Rj. $f(x,y)=\sqrt[3]{x+\sqrt[4]{y}}$

$(x_0,y_0)=(6,16) \Rightarrow \Delta x=x-x_0=-0.2$

$\Delta y=y-y_0=0.01$

$f(x_0,y_0)=\sqrt[3]{6+\sqrt[4]{16}}=\sqrt[3]{8}=2$

$\frac{\partial f}{\partial x}(x_0,y_0)=\frac{1}{3}\left(x+\sqrt[4]{y}\right)^{-\frac{2}{3}}\Big|_{(x_0,y_0)}=\frac{1}{3}(6+2)^{-\frac{2}{3}}=$

$=\frac{1}{3}\ \frac{1}{8^{\frac{2}{3}}}=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$

$\frac{\partial f}{\partial y}(x_0,y_0)=\frac{1}{3}\left(x+\sqrt[4]{y}\right)^{-\frac{2}{3}}\cdot\frac{1}{4}y^{-\frac{3}{4}}\Big|_{(x_0,y_0)}=\frac{1}{12}\cdot\frac{1}{4}\cdot\frac{1}{16^{3/4}}=$

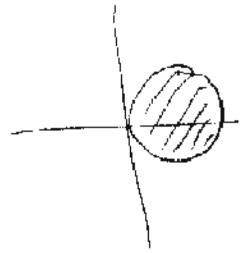
$=\frac{1}{12\cdot 4\cdot 8}$

$f(x,y)\approx 2+\frac{1}{12}\cdot(-0.2)+\frac{1}{48\cdot 8}\cdot 0.01$

4. $\iint\limits_D f(x,y)\,dx\,dy$, D omeđeno sa $x^2+y^2 \le 2x$

$f(x,y)=\sqrt{x^2+y^2}$

Rj. $(x-1)^2+y^2 \le 1$ je D



$r^2 = 2r\cos\varphi$

$r = 2\cos\varphi$

$f(r,\varphi)=\sqrt{r^2}=r$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{3}r^3 \Big|_0^{2\cos\varphi}\right) d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} 8\cos^3\varphi\,d\varphi = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\,d\varphi =$$

$$= \frac{8}{3}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\sin^2\varphi\,d\varphi\right) =$$

$$= \frac{8}{3}\left(\sin\varphi - \frac{1}{3}\sin^3\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = \frac{8}{3}\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right) = \frac{32}{3}$$

5. $y' + \frac{2y}{x} = x^3$

Rj. To je linearna jdn. prvog reda.

Demidović, str. 322, zad 2786

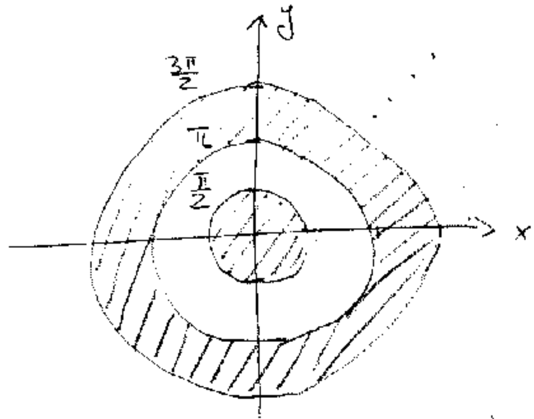
$$y = \frac{1}{6}x^4 + \frac{C}{x^2}$$

(npr metodom varijacije konstanti)

1) Domena od $f = g \circ h$, $h(x,y) = x^2+y^2$, $g(t) = \sqrt{\tan t}$

Rj. $f(x,y) = \sqrt{\tan(x^2+y^2)} \Rightarrow \tan(x^2+y^2) \geq 0$

$\Rightarrow x^2+y^2 \in [0, \frac{\pi}{2}] + 2k\pi,\ k \geq 0$ jer $x^2+y^2 \geq 0$



2) Promijeniti poredak integracije u integralu:

$$\int_0^{8/5} dy \int_{2-\sqrt{4-y^2}}^{\sqrt{2y-y^2}} f(x,y)\,dx = I$$

Rj.

$x_1 = 2-\sqrt{4-y^2} \Rightarrow x_1 - 2 = -\sqrt{4-y^2} \quad /^2$

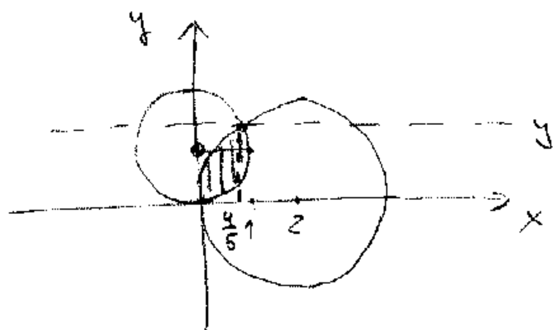
$(x_1-2)^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-(x-2)^2}$

$x_2 = \sqrt{2y-y^2} \Rightarrow x_2^2 + y^2 - 2y = 0 \Rightarrow x_2^2 + (y-1)^2 = 1$

$(y-1)^2 = 1 - x_2^2$

$y = 1 \pm \sqrt{1-x_2^2}$

skica:



stavimo $y = \frac{8}{5} \Rightarrow x_1 - 2 = -\sqrt{4-\frac{64}{25}} \Rightarrow x_1 = -\frac{6}{5}+2 = \frac{4}{5}$

$x_2 = \sqrt{2\cdot\frac{8}{5} - \frac{64}{25}} = \sqrt{\frac{16\cdot 5}{25} - \frac{64}{25}} = \frac{4}{5}$

očito $x_1 = x_2$ pa je to točka presjeka. Sada imamo:

$$I = \int_0^{\frac{4}{5}} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{4-(x-2)^2}} f(x,y)\,dy + \int_{4/5}^{1} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,dy$$

3) Na plohi $z = xy\ln(x^2+xy+y^2)$ odrediti točke u kojima je tang. ravnina okomita na z-os.

Rj Jdn. tang. ravnine u $T_0(x_0, y_0, z_0)$:

$$z - z_0 = f'_x(x_0,y_0)(X-x_0) + f'_y(x_0,y_0)(Y-y_0)$$

ovdje $f(x,y) = xy\ln(x^2+xy+y^2)$

$$f'_x = y\ln(x^2+xy+y^2) + xy\frac{1}{x^2+xy+y^2}(2x+y)$$

$$f'_y = x\ln(x^2+xy+y^2) + xy\frac{1}{x^2+xy+y^2}(2y+x)$$

vektor normale mora biti kolinearan sa vektorom z-osi:

$$f'_x(x_0,y_0)\vec{i} + f'_y(x_0,y_0)\vec{j} - \vec{k} = \lambda\vec{k} \Rightarrow \lambda = -1$$

$$f'_x(x_0,y_0) = 0$$

$$f'_y(x_0,y_0) = 0$$

$$\Rightarrow (x^2+xy+y^2)y\ln(x^2+xy+y^2) + xy(2x+y) = 0$$

$$(x^2+xy+y^2)x\ln(x^2+xy+y^2) + xy(2y+x) = 0$$

$$\Rightarrow \ln(x^2+xy+y^2)(x^2+xy+y^2)(y-x) + xy(\underbrace{2x+y-2y-x}_{x-y}) = 0$$

stavimo $y = x$ i to je očito jedno rješenje.

$$\Rightarrow 3x^3\ln(3x^2) + 3x^3 = 0 \Rightarrow x^3(\ln 3x^2 + 1) = 0$$

$\Rightarrow x = 0$ jedna mogućnost ali onda imamo u $z = xy\ln(x^2+xy+y^2)$ $\ln 0$ što ne da.

$$\Rightarrow \ln(3x^2) + 1 = 0 \Rightarrow 3x^2 = e^{-1} \Rightarrow x^2 = \frac{1}{3e}$$

$$y = x = \pm\sqrt{\frac{1}{3e}}$$

Točke su $x = y = \sqrt{\frac{1}{3e}} \Rightarrow z = \frac{1}{3e}\ln\frac{1}{e} = -\frac{1}{3e}$

$$x = -\sqrt{\frac{1}{3e}},\ y = -\sqrt{\frac{1}{3e}} \Rightarrow z = -\frac{1}{3e}$$

$$T_1\left(\sqrt{\frac{1}{3e}}, \sqrt{\frac{1}{3e}}, -\frac{1}{3e}\right),\ T_2\left(-\sqrt{\frac{1}{3e}}, -\sqrt{\frac{1}{3e}}, -\frac{1}{3e}\right)$$

Ako $x \neq y \Rightarrow \ln(x^2+xy+y^2)(x^2+xy+y^2) = xy$

$$\Rightarrow xy \cdot y + xy(2x+y) = 0 \Rightarrow xy^2 + 2x^2y + xy^2 = 0$$

$$xy \cdot x + xy(2y+x) = 0 \qquad x^2y + 2xy^2 + x^2y = 0$$

$$y \neq 0 \Rightarrow xy + 2x^2 + xy = 0 \Rightarrow 2x^2 + 2xy = 0 \Rightarrow x^2 + xy = 0$$

$\Rightarrow x(x+y)=0$

Ako $x=0 \Rightarrow (\ln y^2)\cdot y^2=0$

$y\neq 0$ po pretpostavci pa $\ln y^2=0 \Rightarrow y^2=1,\ y=\pm$

Ako $x\neq 0 \Rightarrow x+y=0 \Rightarrow x=-y$

$$\Rightarrow \ln(x^2-x^2+x^2)(x^2-x^2+x^2)=-x^2$$

$$(\ln x^2)\,x^2=-x^2$$

$$x^2(1+\ln x^2)=0$$

Jer $x\neq 0 \Rightarrow 1+\ln x^2=0 \Rightarrow \ln x^2=-1 \Rightarrow x^2=\frac{1}{e}$

$$x=\pm\sqrt{\frac{1}{e}}$$

Neka sad $y=0 \Rightarrow (\ln x^2)\,x^2=0$

x mora biti različit od nule $\Rightarrow \ln x^2=0 \Rightarrow x^2=1$

Sve skupa, točke su

$x=0,\ y=\pm 1 \Rightarrow z=0 \Rightarrow T_3(0,1,0),\ T_4(0,-1,0)$

$x=\pm\sqrt{\frac{1}{e}},\ y=\mp\sqrt{\frac{1}{e}} \Rightarrow z=-\frac{1}{e}(-1)=\frac{1}{e}$

$$\Rightarrow T_5\left(\sqrt{\tfrac{1}{e}},-\sqrt{\tfrac{1}{e}},\tfrac{1}{e}\right),\ T_6\left(-\sqrt{\tfrac{1}{e}},\sqrt{\tfrac{1}{e}},\tfrac{1}{e}\right)$$

$y=0,\ x=\pm 1 \Rightarrow$ zo $T_7(1,0,0),\ T_8(-1,0,0)$

4) Taylorov red oko $x=1$, $f(x)=\frac{1}{3-x}$, $f^{(100)}(1)$, konverg.

$$f(x)=\frac{1}{3-x}=\frac{1}{2+1-x}=\frac{1}{2-(x-1)}=\frac{1}{2}\,\frac{1}{1-\frac{x-1}{2}}=$$

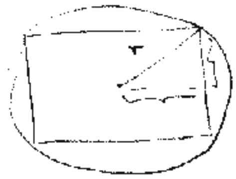
$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(x-1)^n$$

konvergencija: $\left|\frac{x-1}{2}\right|<1 \Rightarrow |x-1|<2 \Rightarrow x\,(-1,3)$

$$\frac{f^{(100)}(1)}{100!}=\frac{1}{2^{101}} \Rightarrow f^{(100)}(1)=\frac{100!}{2^{101}}$$

5) Izračunati približno polumjer opisane kružnice pravokutnika stranica $a = 5.2$, $b = 11.9$.

Rj



$$r = f(a,b) = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$f(a_0+\Delta a, b_0+\Delta b) \approx f(a_0,b_0) + f_a'(a_0,b_0)\Delta a + f_b'(a_0,b_0)\Delta b$$

$$a_0 = 5, \quad b_0 = 12$$

$$\Delta a = a - a_0 = 5.2 - 5 = 0.2$$

$$\Delta b = b - b_0 = 11.9 - 12 = -0.1$$

$$f(a_0,b_0) = \sqrt{\left(\frac{5}{2}\right)^2 + 6^2} = \sqrt{\frac{25}{4} + \frac{4\cdot 36}{4}} = \sqrt{\frac{169}{4}} = \frac{13}{2}$$

$$f_a'(a,b) = \frac{1}{2\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}} \cdot \frac{a}{2} = \frac{1}{4} \frac{a}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}$$

$$f_b'(a,b) = \ldots = \frac{1}{4} \frac{b}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}$$

$$f_a'(a_0,b_0) = \frac{1}{\cancel{4}_2} \cdot \frac{5}{\frac{13}{\cancel{2}}} = \frac{5}{26}, \quad f_b'(a_0,b_0) = \frac{1}{\cancel{4}} \cdot \frac{\cancel{12}^{3}}{\frac{13}{\cancel{2}}} = \frac{6}{13}$$

$$f(5.2, 11.9) = \frac{13}{2} + \frac{5}{26} \cdot 0.2 - \frac{6}{13} 0.1$$

1 Odrediti domenu od $f(x,y) = \arccos\left(1 - \left(\frac{x-y}{x+y}\right)^2\right)$

Rj. zbog arccos: $-1 \le 1 - \left(\frac{x-y}{x+y}\right)^2 \le 1$

$$-2 \le -\left(\frac{x-y}{x+y}\right)^2 \quad i \quad \underbrace{-\left(\frac{x-y}{x+y}\right)^2 \le 0}_{\text{vrijedi uvijek}}$$

$$\Rightarrow 2 \ge \left(\frac{x-y}{x+y}\right)^2$$

$$-\sqrt{2} \le \frac{x-y}{x+y} \le \sqrt{2}$$

a) $x+y > 0$

$$-\sqrt{2}(x+y) \le x-y \le \sqrt{2}(x+y)$$

$$-\sqrt{2}x - \sqrt{2}y \le x - y$$

$$y(1-\sqrt{2}) \le (1+\sqrt{2})x$$

$$y \ge \frac{1+\sqrt{2}}{1-\sqrt{2}}x \approx -5.83x$$

$$x - y \le \sqrt{2}x + \sqrt{2}y$$

$$(1-\sqrt{2})x \le (1+\sqrt{2})y$$

$$-0.17x \approx \frac{1-\sqrt{2}}{1+\sqrt{2}}x \le y$$

b) $x+y < 0$

$$-\sqrt{2}(x+y) \ge x-y \ge \sqrt{2}(x+y)$$

$$-\sqrt{2}x - \sqrt{2}y \ge x - y$$

$$y(1-\sqrt{2}) \ge (1+\sqrt{2})x$$

$$y \le \frac{1+\sqrt{2}}{1-\sqrt{2}}x \approx -5.83x$$

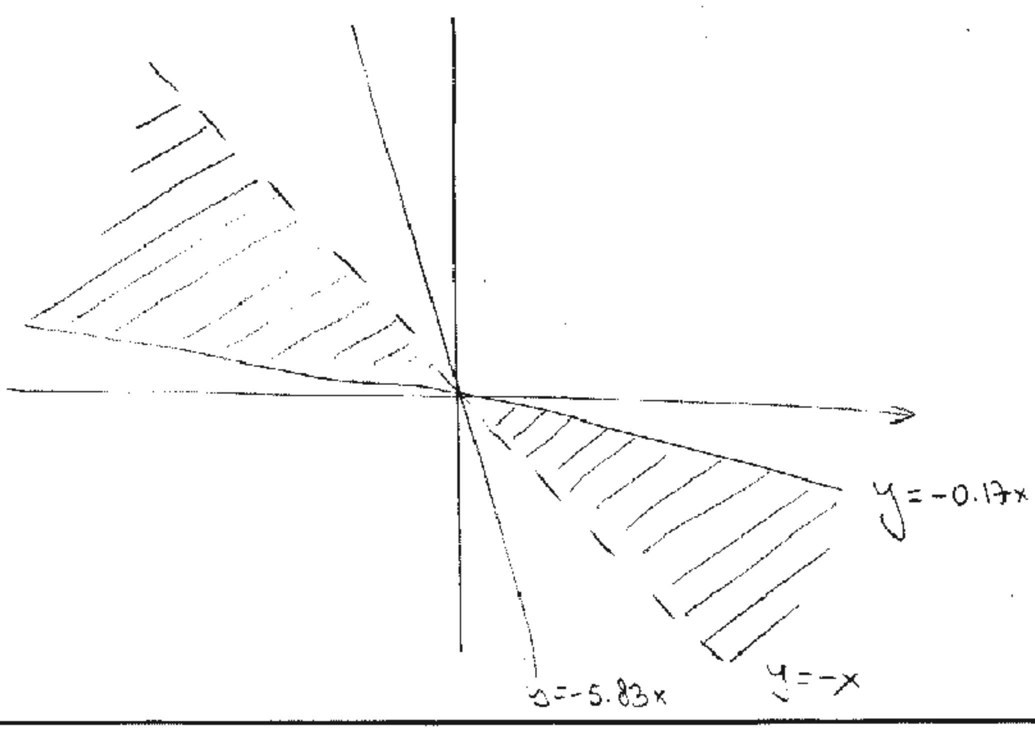
$$x - y \ge \sqrt{2}x + \sqrt{2}y$$

$$(1-\sqrt{2})x \ge (1+\sqrt{2})y$$

$$y \ge \frac{1-\sqrt{2}}{1+\sqrt{2}}x \approx -0.17x$$

$$\frac{1-\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{(1-\sqrt{2})^2}{1-2} = -(1-2\sqrt{2}+2) = -3+2\sqrt{2} \approx -0.17$$

$$\frac{1+\sqrt{2}}{1-\sqrt{2}} \cdot \frac{1+\sqrt{2}}{1+\sqrt{2}} = -(1+\sqrt{2})^2 = -(3+2\sqrt{2}) \approx -5.83$$



3. Na plohi $y^2-x^2-z^2=0$ odredite točku najbližu točki $T(1,0,4)$.

Rj. $M(x_0,y_0,z_0)$ na plohi $\Rightarrow y_0^2-x_0^2-z_0^2=0$

$$y_0^2=x_0^2+z_0^2$$

$$d(M,T)=\sqrt{(x_0-1)^2+y_0^2+(z_0-4)^2}$$

Možemo tražiti minimum i od $d^2(M,T)=(x_0-1)^2+y_0^2+(z_0-4)^2$

$$d^2(M,T)=(x_0-1)^2+x_0^2+z_0^2+(z_0-4)^2 \quad \text{na plohi}$$

$$\frac{\partial d^2}{\partial x}(x_0,z_0)=2(x_0-1)+2x_0=0 \Rightarrow 4x_0=2,\ x_0=\frac{1}{2}$$

$$\frac{\partial d^2}{\partial z}(x_0,z_0)=2z_0+2(z_0-4)=0 \Rightarrow 4z_0=8,\ z_0=2$$

$$y_0^2=\frac{1}{4}+4=\frac{1}{4}+\frac{16}{4}=\frac{17}{4}$$

$$\Rightarrow M_1\left(\frac{1}{2},\sqrt{\frac{17}{4}},2\right),\ M_2\left(\frac{1}{2},-\sqrt{\frac{17}{4}},2\right)$$

4. Ispitajte konv. $\sum_{n=1}^{\infty} a_n$ ako je $a_n=\sqrt{n^2-n}-n$

Rj. nužan uvjet za konvergenciju: $\lim_{n\to\infty} a_n=0$

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\left(\sqrt{n^2-n}-n\right)\cdot\frac{\sqrt{n^2-n}+n}{\sqrt{n^2-n}+n}=\lim_{n\to\infty}\frac{n^2-n-n^2}{\sqrt{n^2-n}+n}=$$

$$=\lim_{n\to\infty}-\frac{n}{\sqrt{n^2-n}+n}=-\frac{1}{2} \quad \text{pa niz ne konv.}$$

5. $y''+5y'+6y=2x^2+5+e^x$ t.d. $y(0)=\frac{1}{30}$, $y'(0)=\frac{1}{10}$

Rj. Homogena jdn: $y''+5y'+6=0$

$$\lambda^2+5\lambda+6=0 \qquad \lambda^2+3\lambda+2\lambda+6=0$$

$$\lambda(\lambda+3)+2(\lambda+3)=0$$

$$(\lambda+3)(\lambda+2)=0$$

$$\lambda_1=-3,\ \lambda_2=-2$$

$$Y(x)=C_1e^{-2x}+C_2e^{-3x}$$

$$f_1(x)=2x^2+5 \Rightarrow a=0$$

$$Y_1=Ax^2+Bx+C$$

$$Y_1'=2Ax+B,\ Y_1''=2A$$

$$2A+5(2Ax+B)+6(Ax^2+Bx+C)=2x^2+5$$

$$2A+\underline{10Ax}+5B+\underline{6Ax^2}+\underline{6Bx}+6C=2x^2+5$$

$6A=2 \Rightarrow A=\frac{1}{3}$

$10A+6B=0 \Rightarrow \frac{10}{3}=-6B \Rightarrow -\frac{1}{\cancel{6}_3}\cdot\frac{\cancel{10}^5}{3}=B=-\frac{5}{9}$

$2A+5B+6C=5 \Rightarrow \frac{2}{3}-\frac{25}{9}+6C=\frac{45}{9}$

$6C=\frac{45}{9}+\frac{25}{9}+\frac{6}{9}=\frac{76}{9}$

$C=\frac{\cancel{76}^{38}}{9}\cdot\frac{1}{\cancel{6}_3}=\frac{38}{27}$

$f_2(x)=e^x$

$\Rightarrow a=1,\ P_n(x)=1 \Rightarrow Y_2(x)=e^x\cdot Q$

$Y_2'(x)=Qe^x,\ Y_2''(x)=Qe^x$

$Qe^x+5Qe^x+6Qe^x=e^x$

$12Q=1 \Rightarrow Q=\frac{1}{12}$

$Y=C_1e^{-2x}+C_2e^{-3x}+\frac{1}{3}x^2-\frac{5}{9}x+\frac{38}{27}+\frac{1}{12}e^x$

$Y(0)=C_1+C_2+\frac{38}{27}+\frac{1}{12}=\frac{1}{30}$

$Y'=-2C_1e^{-2x}-3C_2e^{-3x}+\frac{2}{3}x-\frac{5}{9}+\frac{1}{12}e^x$

$Y'(0)=-2C_1-3C_2-\frac{5}{9}=\frac{1}{10}$ i taj sustav se riješi.

MAT II - 03.10.2005 - RJEŠENJA

1. Odredite domenu fkcije $f(x,y) = \arcsin\left(1-\left(\frac{x}{y}\right)^2\right)$

Rj zbog arcsin: $-1 \le 1-\left(\frac{x}{y}\right)^2 \le 1$

$-2 \le -\left(\frac{x}{y}\right)^2$ i $-\left(\frac{x}{y}\right)^2 \le 0$ što vrijedi uvijek

$2 \geq \left(\frac{x}{y}\right)^2$

$\Rightarrow \frac{x}{y} \in [-\sqrt{2}, \sqrt{2}]$

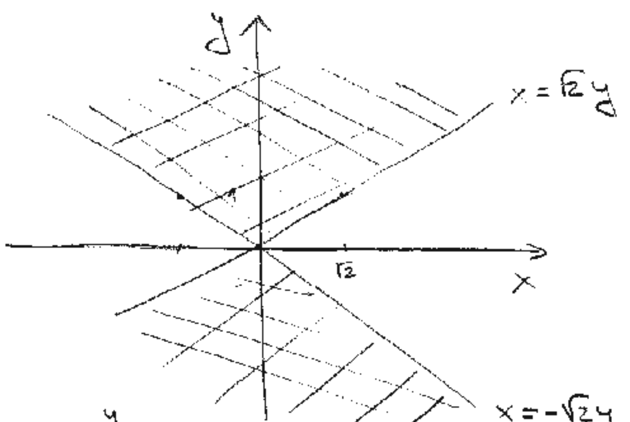
$-\sqrt{2} \le \frac{x}{y}$ i $\frac{x}{y} \le \sqrt{2}$

$y>0$ $y<0$ $y>0$ $y<0$

$-\sqrt{2}\cdot y \le x$ $-\sqrt{2}\cdot y \geq x$ $x \le \sqrt{2}y$ $x \geq \sqrt{2}y$

$\Rightarrow$ $\boxed{y>0}$ $-\sqrt{2}y \le x \le \sqrt{2}y$

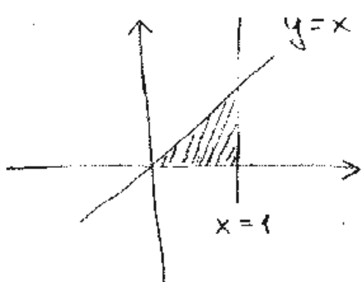
$\boxed{y<0}$ $-\sqrt{2}y \geq x \geq \sqrt{2}y$



2. Izračunati $\iint\limits_{(D)} e^{\frac{y}{x}}\,dx\,dy$, D je područje omeđeno krivuljama

$y = x$, $x = 1$, $y = 0$

Rj skiciramo D:



$$\iint\limits_{(D)} e^{\frac{y}{x}}\,dx\,dy = \int_0^1 dx \int_0^x e^{\frac{y}{x}}\,dy =$$

$$= \int_0^1 \left(x e^{\frac{y}{x}} \Big|_0^x \right) dx = \int_0^1 (xe - 1)\,dx = \frac{e}{2}x^2 - x \Big|_0^1 =$$

$$= \frac{1}{2}e - 0 = \frac{e}{2}$$

<u>4.</u> Ispitajte konverg. $\sum_{n=1}^{\infty} a_n$, $a_n = \frac{1}{3^n - n^2}$

<u>Rj</u> kriterij uspoređivanja: $b_n = \frac{1}{3^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3^n-n^2}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n-n^2} = \lim_{n\to\infty}\left(1+\frac{n^2}{3^n-n^2}\right.$$

$$= 1 \neq 0 \quad \text{i jer } \sum_{n=1}^{\infty} b_n = \frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^n}+\ldots =$$

$$= \frac{1}{3}\left(1+\frac{1}{3}+\ldots\right) = \frac{1}{3}\,\frac{1}{1-\frac{1}{3}}$$ konv., konv. i $\sum_{n=1}^{\infty} a_n$.

<u>5.</u> $\frac{dy}{dx}$, ako je $1+y^x = y$

<u>Rj</u> $y^x = y-1 \quad / \ln$

$x \ln y = \ln(y-1)$ i sada deriviramo po x

$$\ln y + x\frac{1}{y}\cdot y' = \frac{1}{y-1}\cdot y' \Rightarrow y'\left(\frac{1}{y-1}-\frac{x}{y}\right) = \ln y$$

$$y'\,\frac{y-x(y-1)}{y(y-1)} = \ln y$$

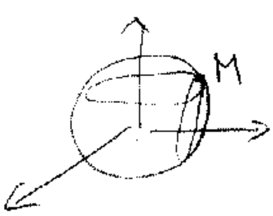
$$\Rightarrow y' = \ln y\,\frac{y(y-1)}{y-x(y-1)} =$$

$$= \ln y\,\frac{y-1}{1-\frac{x}{y}(y-1)} = \ln y\,\frac{y^x}{1-xy^{x-1}}$$

<u>3.</u> Točkom $M(12,4,3)$ kugle $x^2+y^2+z^2=169$ položene su ravn. $\perp$ na os OX i OZ. Nađite ravninu koja prolazi tang. na presječnice u M.

<u>Rj</u> $\perp$ na $OX \Rightarrow \vec{n_1} = \vec{i} \Rightarrow x=12 \ldots \pi_1$ $\quad y^2+z^2=25$

$\perp$ na $OZ \Rightarrow \vec{n_2} = \vec{k} \Rightarrow z=3 \ldots \pi_2$ $\quad x^2+y^2=160$



Tangenta na $y^2+z^2=25$ kružnicu: $x=12$,

$2y_0(y-4)+2z_0(z-3)=0 \Rightarrow 4y-16+3z-9=0$

$3z = 25-4$

$x=12$

$y=t$

$z=\frac{25}{3}-\frac{4}{3}t$

$\Rightarrow \vec{s_1} = \vec{j}-\frac{4}{3}$

ili $\vec{s_1} = 3\vec{j}-4$

Analogno druga: $x=s$

$y=40-3s$

$z=3$

$\Rightarrow \vec{s_2} = \vec{i}-3\vec{j}$

Ravnina: $\vec{n} = \vec{s_1}\times\vec{s_2} =$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -4 \\ 1 & -3 & 0 \end{vmatrix} = -12\vec{i}-4\vec{j}-3\vec{k},$$ M točka pa: $12(x-12)+4(y-4)+3(z-3)=$

MAT II /12.11.2005/ RJEŠENJA

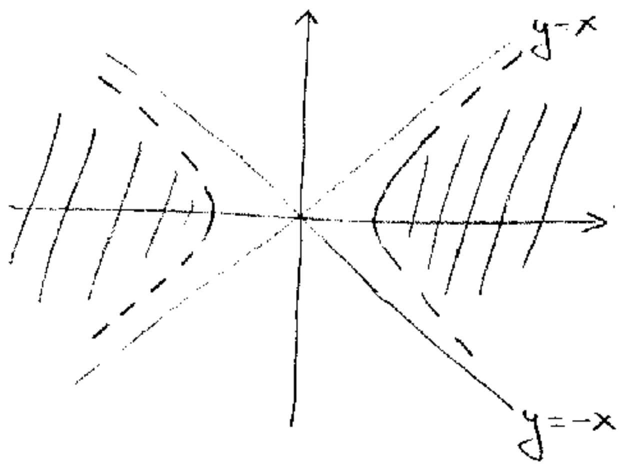
1. $D(f)$, $f(x,y) = g \circ h$, $h(x,y) = x^2 - y^2 - 1$, $g(t) = \ln t^2 + \sqrt{t}$

Rj $f(x,y) = \ln (x^2 - y^2 - 1)^2 + \sqrt{x^2 - y^2 - 1}$

zbog $\sqrt{}$: $x^2 - y^2 - 1 > 0$

zbog $\ln$: $x^2 - y^2 - 1 \neq 0$

sve skupa $x^2 > y^2 + 1$. Skiciramo $x^2 - y^2 = 1$



bez hiperbole!

2. Nađite ekstreme fkcije $z(x,y) = x^3 y^2 (6 - x - y)$, $x > 0, y > 0$

Rj $z_x(x,y) = 3x^2 y^2 (6 - x - y) - x^3 y^2$

$z_y(x,y) = 2x^3 y (6 - x - y) - x^3 y^2$

$z_x(T) = z_y(T) = 0 \Rightarrow$ $3x^2 y^2 (6 - x - y) - x^3 y^2 = 0$

$2x^3 y (6 - x - y) - x^3 y^2 = 0$

$3x^2 y^2 (6 - x - y) = x^3 y^2 \quad / x^2 y$

$2x^3 y (6 - x - y) = x^3 y^2 \quad / x^3 y$

$xy \neq 0 \Rightarrow$ $3(6 - x - y) = x$

$2(6 - x - y) = y$

$6 - x - y \neq 0 \Rightarrow \frac{3}{2} = \frac{x}{y} \Rightarrow 3y = 2x \Rightarrow y = \frac{2}{3}x$

Vraćamo to u jednu od jednadžbi:

$2x^3 \frac{2}{3} x \left(6 - x - \frac{2}{3}x\right) - x^3 \frac{4}{9} x^2 = 0 \quad / : x^4$

$\frac{4}{3}\left(6 - \frac{5}{3}x\right) - \frac{4}{9}x = 0$

$\frac{24}{3} - \frac{20}{9}x - \frac{4}{9}x = 0$

$8 - \frac{24}{9}x = 0 \Rightarrow \cancel{8}^{1} = \frac{\cancel{24}^{3}}{9}x$ $\quad x = 3$

$y = \frac{2}{3} \cdot 3 = 2$

prvi kandidat $T(3,2)$

$xy=0 \Rightarrow$ a) $x=0$ ili b) $y=0$ što nije
$6-x-y=0 \Rightarrow x^3y^2=0$ pa se to svodi pod a) ili b)

$$z_{xx}(x,y)=6xy^2(6-x-y)-3x^2y^2-3x^2y^2$$
$$=6xy^2(6-x-y)-6x^2y^2$$
$$z_{yy}(x,y)=2x^3(6-x-y)-2x^3y-2x^3y=$$
$$=2x^3(6-x-y)-4x^3y$$
$$z_{xy}(x,y)=6x^2y(6-x-y)-3x^2y^2-2x^3y$$

$T(3,2)$

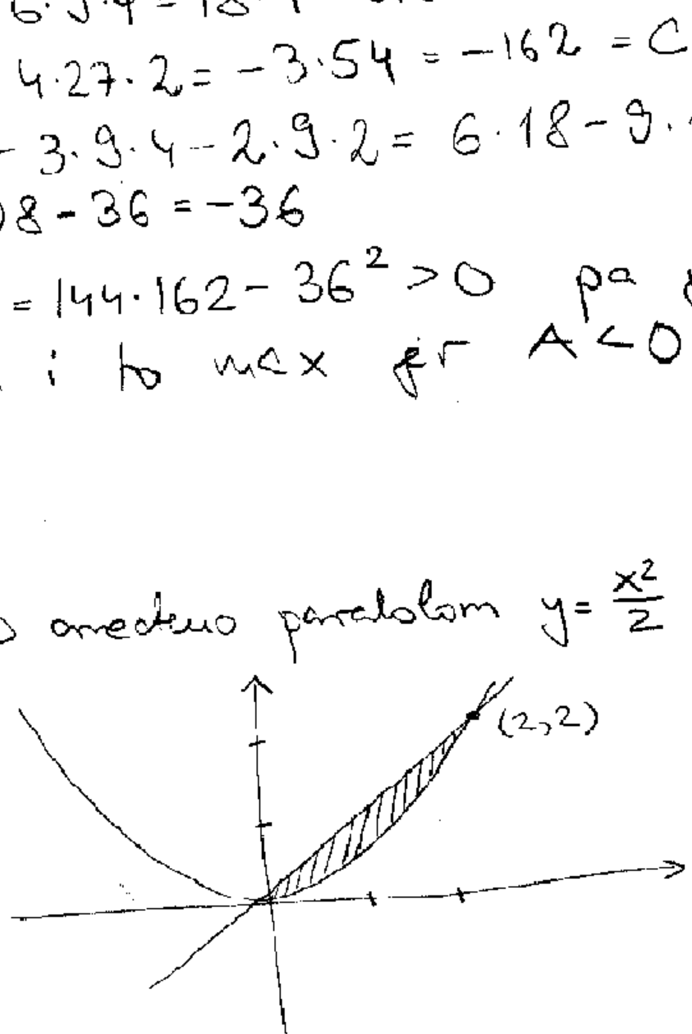
$$z_{xx}(3,2)=6\cdot3\cdot4-6\cdot9\cdot4=18\cdot4-216=-144=A$$
$$z_{yy}(3,2)=2\cdot27-4\cdot27\cdot2=-3\cdot54=-162=C$$
$$z_{xy}(3,2)=6\cdot9\cdot2-3\cdot9\cdot4-2\cdot9\cdot2=6\cdot18-9\cdot12-36$$
$$=108-108-36=-36$$

$$D=AC-B^2=144\cdot162-36^2>0$$ pa je tu lokalni ekstrem i to max jer $A<0$.

b) isto tako

3. $\iint\limits_D \frac{x}{x^2+y^2}\,dx\,dy$, D omeđeno parabolom $y=\frac{x^2}{2}$ i $y=x$

Rj. skiciramo D:



presjecišta: $x=\frac{x^2}{2} \Rightarrow T_1(0,0)$
$1=\frac{x}{2} \Rightarrow T_2=(2,2)$

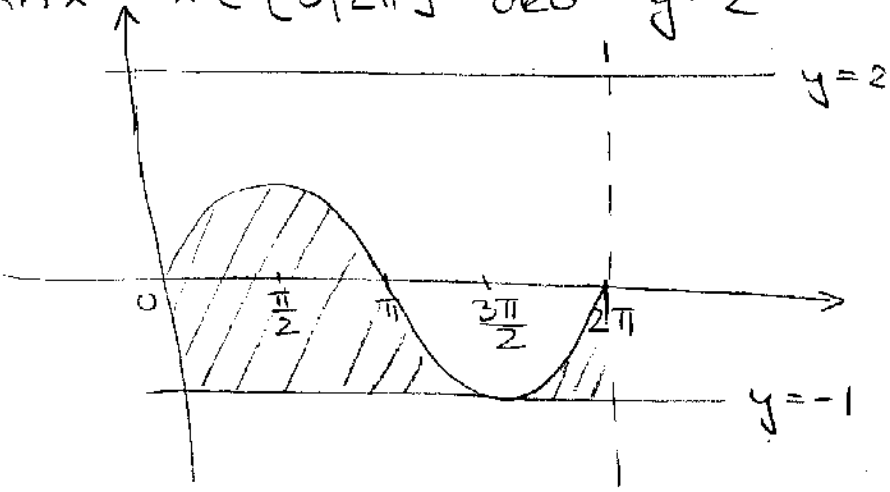
$$\iint\limits_D \frac{x}{x^2+y^2}\,dx\,dy=\int_0^2 dx\int_{x^2/2}^{x}\frac{x}{x^2+y^2}\,dy=\int_0^2 x\,dx\int_{x^2/2}^{x}\frac{dy}{x^2+y^2}=$$
$$=\int_0^2 x\left(\frac{1}{x}\operatorname{arctg}\frac{y}{x}\Big|_{x^2/2}^{x}\right)dy=\int_0^2\left(\operatorname{arctg}1-\operatorname{arctg}\frac{x}{2}\right)dx=$$

$= \frac{\pi}{4} x - \left(x \operatorname{arctg} \frac{x}{2} - \ln\left(1+\left(\frac{x}{2}\right)^2\right)\right) \Big|_0^2 = \frac{\pi}{2} -$

$-\left(2 \cdot \frac{\pi}{4} - \ln 2\right) = \ln 2$

4. Izračunajte obujam tijela nastalog rotacijom $-1 \le y \le \sin x$, $x \in [0, 2\pi]$ oko $y = 2$

Rj.



radimo translaciju po y osi, $y \to y - 2 = t$

$-1 \le t + 2 \le \sin x$

$-3 \le t \le \sin x - 2$

os rotacije: $t + 2 = 2$ tj. $t = 0$ tj. x-os. Sve slupa

$$V = V_1 - V_2 = \pi \int_0^{2\pi} (-3)^2 dx - \pi \int_0^{2\pi} (\sin x - 2)^2 dx =$$

$$= \pi \int_0^{2\pi} (9 - \sin^2 x + 4\sin x - 4) dx = \pi \int_0^{2\pi} (5 - \sin^2 x + 4\sin x) dx$$

$$= \pi \int_0^{2\pi} \left(5 - \frac{1-\cos 2x}{2} + 4\sin x\right) dx = \pi \int_0^{2\pi} \left(\frac{9}{2} + \frac{\cos 2x}{2} + 4\sin x\right) dx$$

$$= \pi \left(\frac{9}{2}x + \frac{1}{4}\sin 2x - 4\cos x \Big|_0^{2\pi}\right) =$$

$$= \pi (9\pi - 4 + 4) = 9\pi^2$$

5. Nađite partikularno rješenje jednadžbe $(1+e^x) y y' = e^x$ ako je $y(0) = 1$.

Rj. $(1+e^x) y \frac{dy}{dx} = e^x \Rightarrow y\,dy = \frac{e^x}{1+e^x} dx \quad / \int$

$\int y\,dy = \int \frac{e^x}{1+e^x} dx = \frac{1}{2} y^2 = \ln(1+e^x) + C \Rightarrow$

$y^2 = 2\ln(1+e^x) + C$, $1 = \underbrace{2 \cdot \ln 1}_{=0} + C \Rightarrow C = 1$

$y^2 = 2\ln(1+e^x) + 1$