

1. (i)

$$\int_{\widehat{AB}} Pdx + Qdy + Rdz = \int_{t_1}^{t_2} [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]dt,$$

gdje su:

$P(x, y, z), Q(x, y, z), R(x, y, z)$ odgovarajuće funkcije triju varijabla,

\widehat{AB} orijentirani luk prostorne krivulje $t \mapsto (x(t), y(t), z(t))$,

$A = (x(t_1), y(t_1), z(t_1)), B = (x(t_2), y(t_2), z(t_2))$,

$x'(t) = \frac{dx(t)}{dt}, y'(t) = \frac{dy(t)}{dt}, z'(t) = \frac{dz(t)}{dt}$.

(ii) Tu je:

$t_1 = 0, t_2 = \pi$,

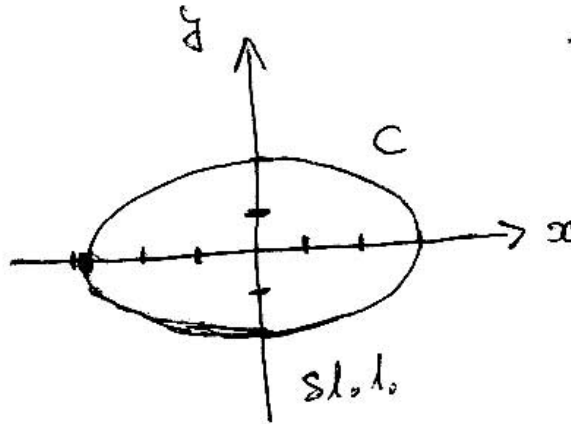
$P(x, y, z) = y, Q(x, y, z) = -x, R(x, y, z) = z$,

$x(t) = 2 \cos t, y(t) = 2 \sin t, z(t) = 3t$,

$x'(t) = -2 \sin t, y'(t) = 2 \cos t, z'(t) = 3$.

$$\begin{aligned} \int_{\widehat{AB}} ydx - xdy + zdz &= \int_0^\pi [2 \sin t(-2 \sin t) - 2 \cos t \cdot 2 \cos t + 3t \cdot 3]dt = \\ &= \int_0^\pi [-4(\sin^2 t + \cos^2 t) + 9t]dt = \int_0^\pi [-4 + 9t]dt = (-4t + \frac{9t^2}{2})|_0^\pi = -4\pi + \frac{9\pi^2}{2}. \end{aligned}$$

Varijanta: tu je (sl.1.)



$x(t) = 3 \cos t, y(t) = 2 \sin t$

$x'(t) = -3 \sin t, y'(t) = 2 \cos t$

$t_1 = 0, t_2 = 2\pi$

$P(x, y) = x + y, Q(x, y) = x - y$ (varijable z nema).

$$\begin{aligned} \oint_C (x + y)dx + (x - y)dy &= \int_0^{2\pi} [(3 \cos t + 2 \sin t)(-3 \sin t) + (3 \cos t - 2 \sin t)2 \cos t]dt = \\ &= \int_0^{2\pi} [-13 \sin t \cos t + 6(\cos^2 t - \sin^2 t)]dt = \int_0^{2\pi} [\frac{-13}{2} \sin 2t + 6 \cos 2t]dt = 0. \end{aligned}$$

2. (i)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{\Gamma} P dx + Q dy,$$

gdje je D područje u ravnini koje obrubljuje pozitivno orijentirana zatvorena krivulja Γ koja nema točaka samopresijecanja, a $P(x, y)$, $Q(x, y)$ su odgovarajuće funkcije dviju varijabla.

(ii)

$$\oint_{\Gamma} x dx + y dy = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) dx dy = \iint_D (0 - 0) \cdot dx dy = 0.$$

Ovdje je D kvadrat koji obrubljuje krivulja Γ .

Varijanta:

$$\oint_{\Gamma} y dx - x dy = \iint_D \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) dx dy = \iint_D (-2) dx dy = -2 \cdot 9\pi = -18\pi,$$

jer je površina kruga D jednaka 9π .

3. (i) $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$,

gdje su $a_x(x, y, z)$, $a_y(x, y, z)$, $a_z(x, y, z)$ funkcije triju varijabla.

$$\text{Divergencija: } \text{div} \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\text{Rotacija: } \text{rot}(\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$\text{Nabla operator: } \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$

$\text{div} \mathbf{a} = \nabla \cdot \mathbf{a}$ (skalarni umnožak)

$\text{rota} = \nabla \times \mathbf{a}$ (vektorski umnožak)

(ii) $\text{div} \mathbf{a} = \frac{\partial(x+y^2)}{\partial x} + \frac{\partial(y+z^2)}{\partial y} + \frac{\partial(z+x^2)}{\partial z} = 1 + 1 + 1 = 3$ (konstantna funkcija).

$$\text{rot}(\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \mathbf{i}(0-2z) - \mathbf{j}(2x-0) + \mathbf{k}(0-2y) = -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}$$

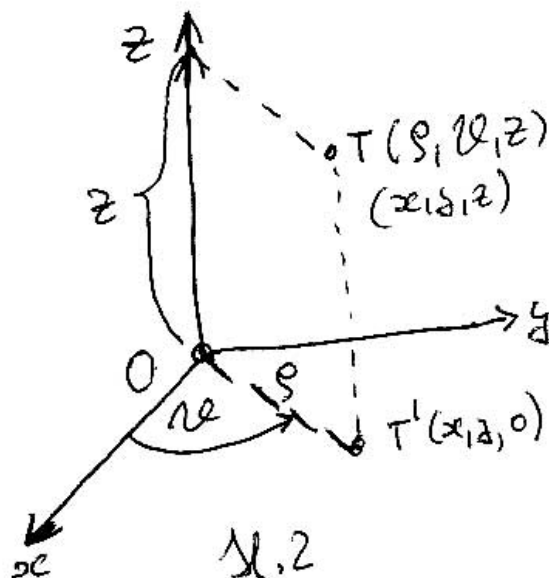
$$\text{div}(\text{rota}) = \text{div}(-2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}) = 0 + 0 + 0 = 0$$

4. (i) Za točku T s kartezijevim koordinatama (x, y, z) koja nije na z -osi (tj. za koju je $x \neq 0$ ili $y \neq 0$) cilindrične koordinate (ρ, θ, z) dobiju se ovako (sl.2.):

ρ je udaljenost projekcije $T'(x, y, 0)$ točke T na $x - y$ ravninu od ishodišta $O(0, 0, 0)$.

θ je kut od pozitivne zrake x -osi do spojnice OT' , krećući se suprotno kazaljki sata.

z je uobičajena treća kartezijeva koordinata (aplikata).



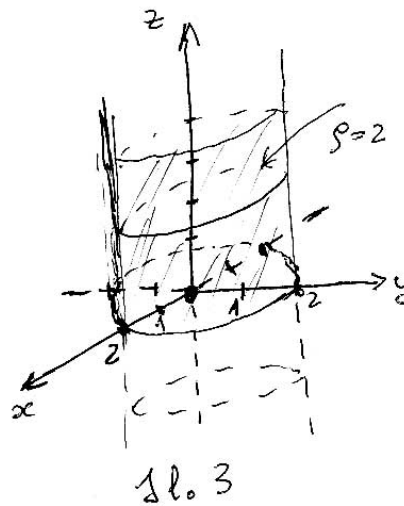
Vrijedi:

$$0 < \rho < +\infty, 0 \leq \theta < 2\pi \text{ (u radijanima)}, -\infty < z < +\infty.$$

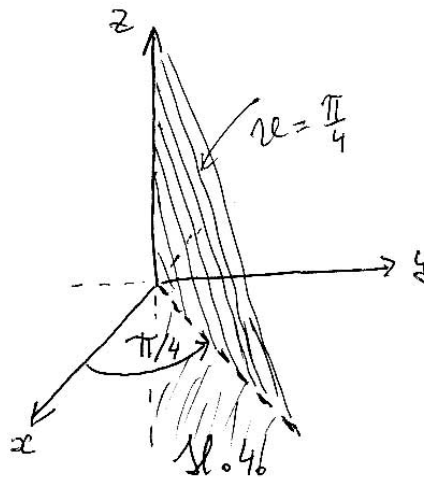
Veza među koordinatama:

$$x = \rho \cos \theta, y = \rho \sin \theta, z = z.$$

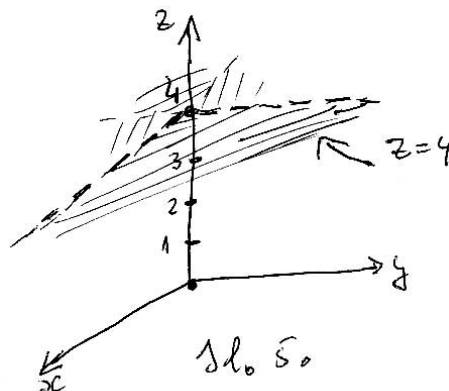
(ii) $\rho = 2$ je jednačba plašta beskonačnog valjka (cilindra) (sl. 3.).



$\theta = \frac{\pi}{4}$ je jednačba poluravnine okomite na $x - y$ ravninu (sl.4.).



$z = 4$ je jednačba ravnine usporodne $x - y$ ravninom na visini 4 (sl.5.).



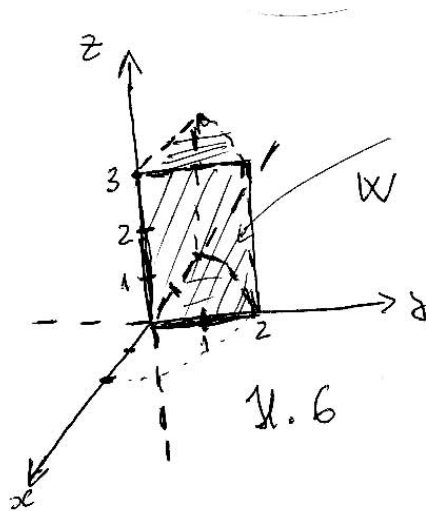
(iii) Diferencijal obujma: $dV = \rho d\rho d\theta dz$.

U cilindričnim koordinatama W je zadan nejednakostima:

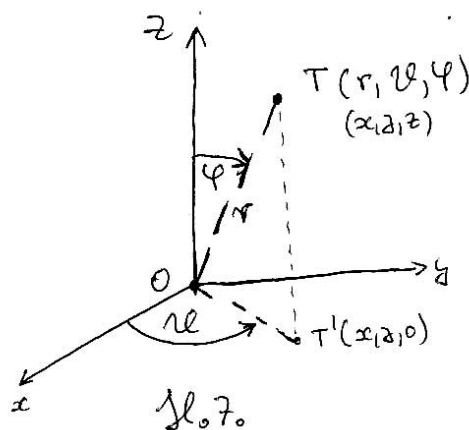
$\frac{\pi}{2} \leq \theta \leq \pi$, $0 < \rho \leq 2$, $0 \leq z \leq 3$ (sl.6.).

Zato je (u posljednjem integralu množimo s ρ):

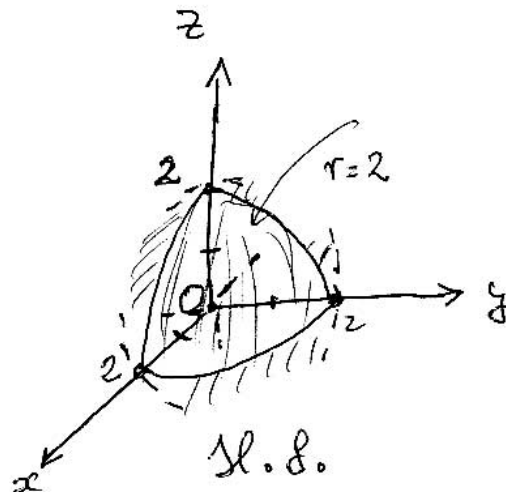
$$\begin{aligned} \iiint_W yz dx dy dz &= \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 d\rho \int_0^3 \rho \sin \theta \cdot z \rho dz = \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 \frac{9}{2} \sin \theta \rho^2 d\rho = \\ &= \int_{\frac{\pi}{2}}^{\pi} \frac{9}{2} \sin \theta \cdot \frac{8}{3} d\theta = (-12 \cos \theta) \Big|_{\frac{\pi}{2}}^{\pi} = -12(-1 - 0) = 12. \end{aligned}$$



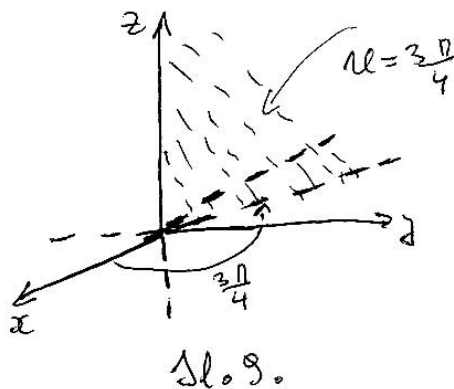
5. (i) Za točku T s kartezijevim koordinatama (x, y, z) koja nije na z -osi (tj. za koju je $x \neq 0$ ili $y \neq 0$) sferne koordinate (r, θ, ϕ) dobiju se ovako (sl.7.):
 r je udaljenost točke T od ishodišta.
 θ je (kao i kod cilindričnih koordinata) kut od pozitivne zrake x -osi do spojnice OT' , krećući se suprotno kazaljki sata.
 ϕ je manji od kutova što ga zatvara pozitivna zraka z -osi sa spojnicom OT .
 Vrijedi: $0 < r < +\infty$, $0 \leq \theta < 2\pi$, $0 < \phi < \pi$.
 Veza s kartezijevim koordinatama: $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \phi$.



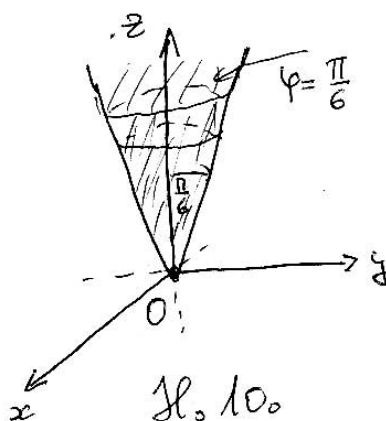
- (ii) $r = 2$ je jednadžba sfere polumjera 2 sa središtem u ishodištu (sl.8.).



$\theta = \frac{3\pi}{4}$ je (kao i u cilindričnim koordinatama) jednačba poluravnine okomite na $x - y$ ravninu (sl.9).



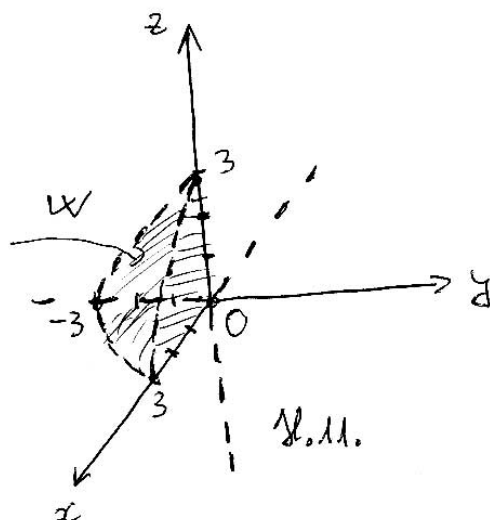
$\phi = \frac{\pi}{6}$ je jednačba plašta beskonačnog stožca (sl.10).



(iii) Diferencijal obujma: $dV = r^2 \sin \phi dr d\theta d\phi$.

U sfernim koordinatama W je zadan nejednakostima (sl.11):

$$\frac{3\pi}{2} \leq \theta \leq 2\pi, \quad 0 < r \leq 3, \quad 0 < \phi \leq \frac{\pi}{2}.$$



Zato je (u zadnjem integralu množimo s $r^2 \sin \phi$):

$$\begin{aligned} \iint_W z dx dy dz &= \int_{\frac{3\pi}{2}}^{2\pi} d\theta \int_0^3 dr \int_0^{\frac{\pi}{2}} r \cos \phi \cdot r^2 \sin \phi d\phi = \\ &= \int_{\frac{3\pi}{2}}^{2\pi} d\theta \int_0^3 \frac{1}{2} r^3 dr = \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2} \cdot \frac{3^4}{4} d\theta = \frac{81}{8} \cdot \frac{\pi}{2} = \frac{81\pi}{16}. \end{aligned}$$

(U drugom smo koraku iskoristili da je $\int_0^{\frac{\pi}{2}} \cos \phi \sin \phi d\phi = \int_0^{\frac{\pi}{2}} \frac{\sin(2\phi)}{2} d\phi = \frac{1}{2}$.)