1. (i)

$$\int_{\overset{\frown}{AB}} Pdx + Qdy + Rdz =$$

$$\int_{t_1}^{t_2} [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]dt,$$

gdje su:

$P(x, y, z), Q(x, y, z), R(x, y, z)$ odgovarajuće funkcije triju varijabla,

$\overset{\curvearrowright}{AB}$ orijentirani luk prostorne krivulje $t \mapsto (x(t), y(t), z(t))$,

$A = (x(t_1), y(t_1), z(t_1)),\ B = (x(t_2), y(t_2), z(t_2))$,

$x'(t) = \frac{dx(t)}{dt},\ y'(t) = \frac{dy(t)}{dt},\ z'(t) = \frac{dz(t)}{dt}$.

(ii) Tu je:

$t_1 = 0,\ t_2 = \pi$,

$P(x, y, z) = y,\ Q(x, y, z) = -x,\ R(x, y, z) = z$,

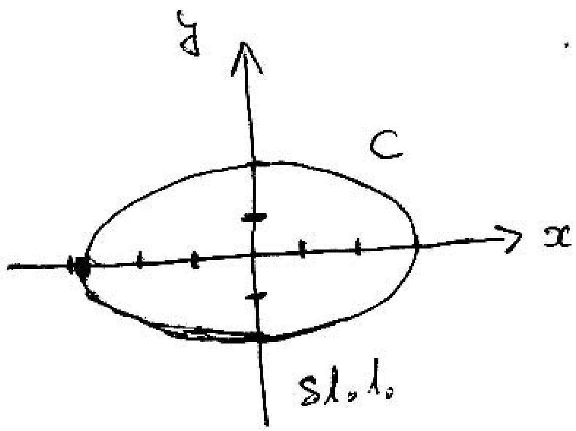
$x(t) = 2\cos t,\ y(t) = 2\sin t,\ z(t) = 3t$,

$x'(t) = -2\sin t,\ y'(t) = 2\cos t,\ z'(t) = 3$.

$$\int_{\overset{\frown}{AB}} ydx - xdy + zdz = \int_0^{\pi} [2\sin t(-2\sin t) - 2\cos t \cdot 2\cos t + 3t \cdot 3]dt =$$

$$= \int_0^{\pi} [-4(\sin^2 t + \cos^2 t) + 9t]dt = \int_0^{\pi} [-4 + 9t]dt = (-4t + \frac{9t^2}{2})|_0^{\pi} = -4\pi + \frac{9\pi^2}{2}.$$

Varijanta: tu je (sl.1.)



sl. 1.

$x(t) = 3\cos t,\ y(t) = 2\sin t$

$x'(t) = -3\sin t,\ y'(t) = 2\cos t$

$t_1 = 0,\ t_2 = 2\pi$

$P(x, y) = x + y,\ Q(x, y) = x - y$ (varijable z nema).

$$\oint_C (x + y)dx + (x - y)dy = \int_0^{2\pi} [(3\cos t + 2\sin t)(-3\sin t) + (3\cos t - 2\sin t)2\cos t]dt =$$

$$= \int_0^{2\pi} [-13\sin t\cos t + 6(\cos^2 t - \sin^2 t)]dt = \int_0^{2\pi} [\frac{-13}{2}\sin 2t + 6\cos 2t)]dt = 0.$$

2. (i)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_\Gamma Pdx + Qdy,$$

gdje je D područje u ravnini koje obrubljuje pozitivno orijentirana zatvorena krivulja Γ koja nema točaka samopresijecanja, a $P(x,y),\ Q(x,y)$ su odgovarajuće funkcije dviju varijabla.

(ii)

$$\oint_\Gamma xdx + ydy = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) dxdy = \iint_D (0-0)\cdot dxdy = 0.$$

Ovdje je D kvadrat koji obrubljuje krivulja Γ.

Varijanta:

$$\oint_\Gamma ydx - xdy = \iint_D \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right) dxdy = \iint_D (-2) dxdy = -2\cdot 9\pi = -18\pi,$$

jer je površina kruga D jednaka 9π.

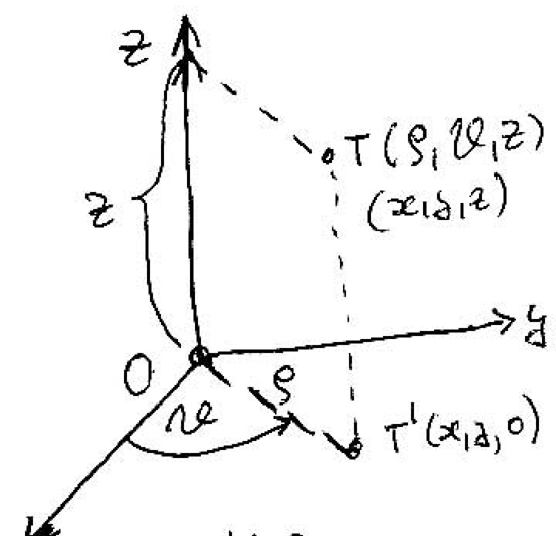
3. (i) $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$,
gdje su $a_x(x,y,z), a_y(x,y,z), a_z(x,y,z)$ funkcije triju varijabla.
Divergencija: $\text{div}\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$
Rotacija: $\text{rot}(\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$
Nabla operator: $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$
$\text{div}\mathbf{a} = \nabla\cdot\mathbf{a}$ (skalarni umnožak)
$\text{rot}\mathbf{a} = \nabla\times\mathbf{a}$ (vektorski umnožak)

(ii) $\text{div}\mathbf{a} = \frac{\partial(x+y^2)}{\partial x} + \frac{\partial(y+z^2)}{\partial y} + \frac{\partial(z+x^2)}{\partial z} = 1+1+1 = 3$ (konstantna funkcija).

$$\text{rot}(\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \mathbf{i}(0-2z) - \mathbf{j}(2x-0) + \mathbf{k}(0-2y) = -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}$$

$\text{div}(\text{rot}\mathbf{a}) = \text{div}(-2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}) = 0+0+0 = 0$

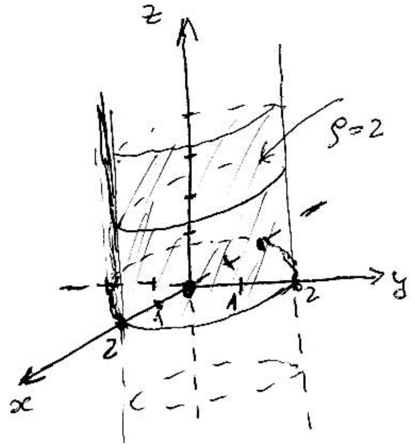
4. (i) Za točku T s kartezijevim koordinatama (x,y,z) koja nije na z-osi (tj. za koju je $x \neq 0$ ili $y \neq 0$) cilindrične koordinate (ρ,θ,z) dobiju se ovako (sl.2.):
ρ je udaljenost projekcije $T'(x,y,0)$ točke T na $x-y$ ravninu od ishodišta $O(0,0,0)$.
θ je kut od pozitivne zrake x-osi do spojnice OT', krećući se suprotno kazaljki sata.
z je uobičajena treća kartezijeva koordinata (aplikata).



sl.2

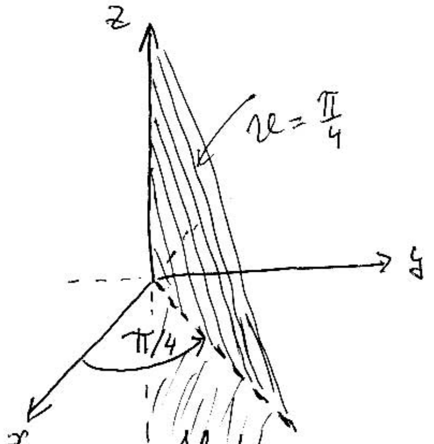
Vrijedi:
$0 < \rho < +\infty,\ 0 \leq \theta < 2\pi$ (u radijanima), $-\infty < z < +\infty$.
Veza među koordinatama:
$x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = z.$

(ii) $\rho = 2$ je jednadžba plašta beskonačnog valjka (cilindra) (sl. 3.).



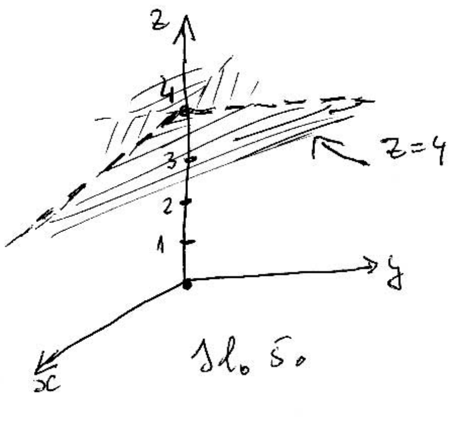
sl. 3

$\theta = \frac{\pi}{4}$ je jednadžba poluravnine okomite na $x - y$ ravninu (sl.4.).



sl. 4.

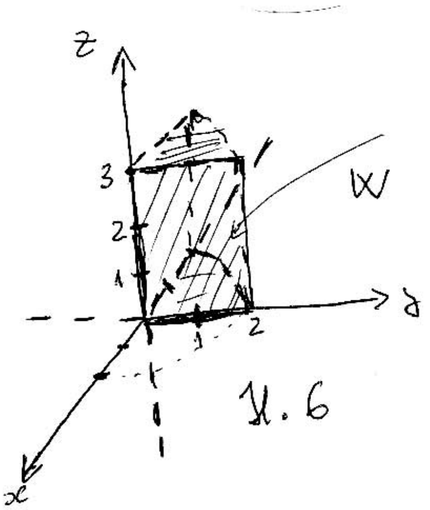
$z = 4$ je jednadžba ravnine usporedne s $x - y$ ravninom na visini 4 (sl.5.).



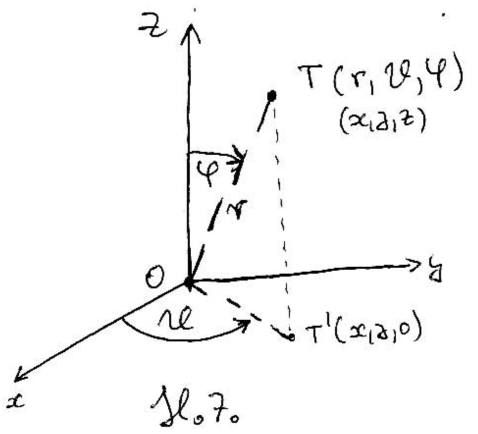
sl. 5.

(iii) Diferencijal obujma: $dV = \rho d\rho d\theta dz$.
U cilindričnim koordinatama W je zadan nejednakostima:
$\frac{\pi}{2} \leq \theta \leq \pi,\ 0 < \rho \leq 2,\ 0 \leq z \leq 3$ (sl.6.).
Zato je (u posljednjem integralu množimo s ρ):

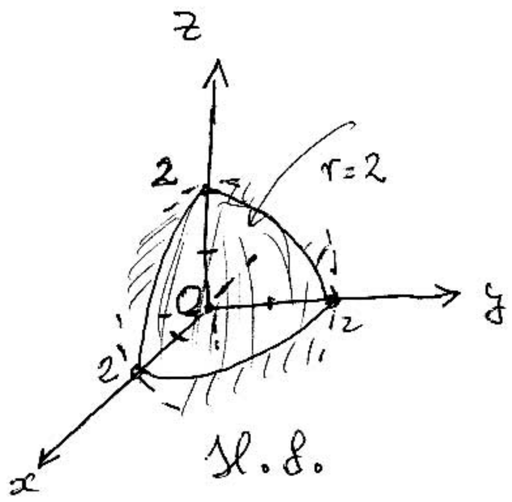
$$\iiint_W yzdxdydz = \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 d\rho \int_0^3 \rho \sin\theta \cdot z\rho dz = \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 \frac{9}{2} \sin\theta \rho^2 d\rho =$$
$$= \int_{\frac{\pi}{2}}^{\pi} \frac{9}{2} \sin\theta \cdot \frac{8}{3} d\theta = (-12\cos\theta)|_{\frac{\pi}{2}}^{\pi} = -12(-1-0) = 12.$$



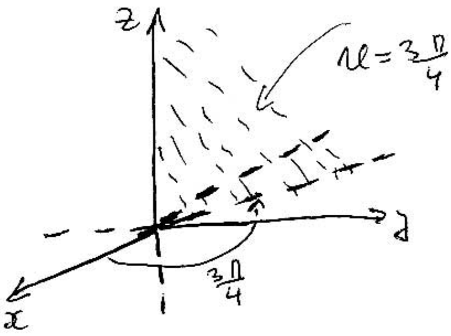
5. (i) Za točku T s kartezijevim koordinatama (x, y, z) koja nije na z-osi (tj. za koju je $x \neq 0$ ili $y \neq 0$) sferne koordinate (r, θ, ϕ) dobiju se ovako (sl.7.):
r je udaljenost točke T od ishodišta.
θ je (kao i kod cilindričnih koordinata) kut od pozitivne zrake x-osi do spojnice OT', krećući se suprotno kazaljki sata.
ϕ je manji od kutova što ga zatvara pozitivna zraka z-osi sa spojnicom OT.
Vrijedi: $0 < r < +\infty,\ 0 \leq \theta < 2\pi,\ 0 < \phi < \pi$.
Veza s kartezijevim koordinatama: $x = r \sin\phi \cos\theta,\ y = r \sin\phi \sin\theta,\ z = r \cos\phi$.



(ii) $r = 2$ je jednadžba sfere polumjera 2 sa središtem u ishodištu (sl.8.).

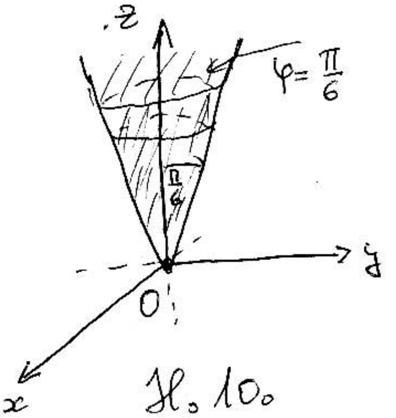


$\theta = \frac{3\pi}{4}$ je (kao i u cilindričnim koordinatama) jednadžba poluravnine okomite na $x - y$ ravninu (sl.9.).



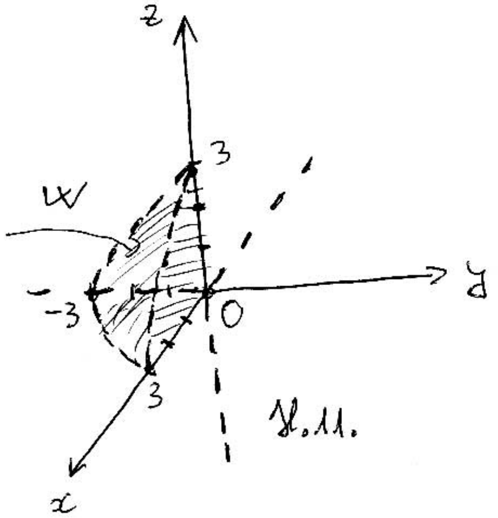
Sl. 9.

$\phi = \frac{\pi}{6}$ je jednadžba plašta beskonačnog stožca (sl.10).



Sl. 10.

(iii) Diferencijal obujma: $dV = r^2 \sin\phi dr d\theta d\phi$.
U sfernim koordinatama W je zadan nejednakostima (sl.11):
$\frac{3\pi}{2} \leq \theta \leq 2\pi,\ 0 < r \leq 3,\ 0 < \phi \leq \frac{\pi}{2}$.



Sl. 11.

Zato je (u zadnjem integralu množimo s $r^2 \sin\phi$):

$$\iint_W z dx dy dz = \int_{\frac{3\pi}{2}}^{2\pi} d\theta \int_0^3 dr \int_0^{\frac{\pi}{2}} r\cos\phi \cdot r^2 \sin\phi d\phi =$$
$$= \int_{\frac{3\pi}{2}}^{2\pi} d\theta \int_0^3 \frac{1}{2} r^3 dr = \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2} \cdot \frac{3^4}{4} d\theta = \frac{81}{8} \cdot \frac{\pi}{2} = \frac{81\pi}{16}.$$

(U drugom smo koraku iskoristili da je $\int_0^{\frac{\pi}{2}} \cos\phi \sin\phi d\phi = \int_0^{\frac{\pi}{2}} \frac{\sin(2\phi)}{2} d\phi = \frac{1}{2}$.)